1

ELECTROSTATICS: CHARGES AND FIELDS

ELECTRIC CHARGE

1.1 Electricity appeared to its early investigators as an extraordinary phenomenon. To draw from bodies the "subtle fire," as it was sometimes called, to bring an object into a highly electrified state, to produce a steady flow of current, called for skillful contrivance. Except for the spectacle of lightning, the ordinary manifestations of nature, from the freezing of water to the growth of a tree, seemed to have no relation to the curious behavior of electrified objects. We know now that electrical forces largely determine the physical and chemical properties of matter over the whole range from atom to living cell. For this understanding we have to thank the scientists of the nineteenth century, Ampère, Faraday, Maxwell, and many others, who discovered the nature of electromagnetism, as well as the physicists and chemists of the twentieth century who unraveled the atomic structure of matter.

Classical electromagnetism deals with electric charges and currents and their interactions as if all the quantities involved could be measured independently, with unlimited precision. Here *classical* means simply "nonquantum." The quantum law with its constant h is ignored in the classical theory of electromagnetism, just as it is in ordinary mechanics. Indeed, the classical theory was brought very nearly to its present state of completion before Planck's discovery. It has survived remarkably well. Neither the revolution of quantum physics nor the development of special relativity dimmed the luster of the electromagnetic field equations Maxwell wrote down 100 years ago.

Of course the theory was solidly based on experiment, and because of that was fairly secure within its original range of application—to coils, capacitors, oscillating currents, and eventually radio waves and light waves. But even so great a success does not guarantee validity in another domain, for instance, the inside of a molecule.

Two facts help to explain the continuing importance in modern physics of the classical description of electromagnetism. First, special relativity required no revision of classical electromagnetism. Historically speaking, special relativity *grew out of* classical electromagnetic theory and experiments inspired by it. Maxwell's field equations, developed long before the work of Lorentz and Einstein, proved to be entirely compatible with relativity. Second, quantum modifications of the electromagnetic forces have turned out to be unimportant down to distances less than 10^{-10} centimeters (cm), 100 times smaller than the atom. We can describe the repulsion and attraction of particles in the atom using the same laws that apply to the leaves of an electroscope, although we need quantum mechanics to predict how the particles will behave under those forces. For still smaller distances, a fusion of electromagnetic theory and quantum theory, called *quantum electrodynamics,* has been remarkably successful. Its predictions are confirmed by experiment down to the smallest distances yet explored.

It is assumed that the reader has some acquaintance with the elementary facts of electricity. We are not going to review all the experiments by which the existence of electric charge was demonstrated, nor shall we review all the evidence for the electrical constitution of matter. On the other hand, we do want to look carefully at the experimental foundations of the basic laws on which all else depends. In this chapter we shall study the physics of stationary electric charges—*electrostatics.*

Certainly one fundamental property of electric charge is its existence in the two varieties that were long ago named *positive* and *negative.* The observed fact is that all charged particles can be divided into two classes such that all members of one class repel each other, while attracting members of the other class. If two small electrically charged bodies *A* and *B*, some distance apart, attract one another, and if *A* attracts some third electrified body *C*, then we always find that *B* repels *C*. Contrast this with gravitation: There is only one kind of gravitational mass, and every mass attracts every other mass.

One may regard the two kinds of charge, positive and negative, as opposite manifestations of one quality, much as *right* and *left* are the two kinds of handedness. Indeed, in the physics of elementary particles, questions involving the sign of the charge are sometimes linked to a question of handedness, and to another basic symmetry, the relation of a sequence of events, *a,* then *b,* then *c,* to the temporally reversed sequence *c,* then *b,* then *a.* It is only the duality of electric charge that concerns us here. For every kind of particle in nature, as far as we know, there can exist an *antiparticle,* a sort of electrical "mirror image." The antiparticle carries charge of the opposite sign. If any other intrinsic quality of the particle has an opposite, the antiparticle has that too, whereas in a property which admits no opposite, such as mass, the antiparticle and particle are exactly alike. The electron's charge is negative; its antiparticle, called a *positron,* has a positive charge, but its mass is precisely the same as that of the electron. The proton's antiparticle is called simply an *antiproton;* its electric charge is negative. An electron and a proton combine to make an ordinary hydrogen atom. A positron and an antiproton could combine in the same way to make an atom of antihydrogen. Given the building blocks, positrons, antiprotons, and antineutrons,† there could be built up the whole range of antimatter, from antihydrogen to antigalaxies. There is a practical difficulty, of course. Should a positron meet an electron or an antiproton meet a proton, that pair of particles will quickly vanish in a burst of radiation. It is therefore not surprising that even positrons and antiprotons, not to speak of antiatoms, are exceedingly rare and short-lived in our world. Perhaps the universe contains,

†Although the electric charge of each is zero, the neutron and its antiparticle are not interchangeable. In certain properties that do not concern us here, they are opposite.

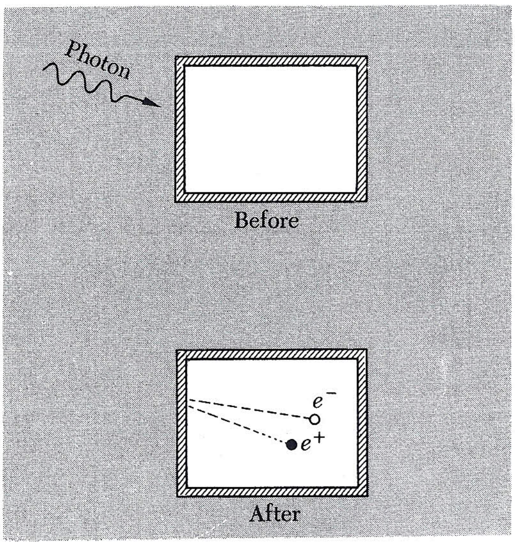
somewhere, a vast concentration of antimatter. If so, its whereabouts is a cosmological mystery.

The universe around us consists overwhelmingly of matter, not antimatter. That is to say, the abundant carriers of negative charge are electrons, and the abundant carriers of positive charge are protons. The proton is nearly 2000 times heavier than the electron and very different, too, in some other respects. Thus matter at the atomic level incorporates negative and positive electricity in quite different ways. The positive charge is all in the atomic nucleus, bound within a massive structure no more than 10^{-12} cm in size, while the negative charge is spread, in effect, through a region about 10^4 times larger in dimensions. It is hard to imagine what atoms and molecules—and all of chemistry—would be like, if not for this fundamental electrical asymmetry of matter.

What we call negative charge, by the way, could just as well have been called positive. The name was a historical accident. There is nothing essentially negative about the charge of an electron. It is not like a negative integer. A negative integer, once multiplication has been defined, differs essentially from a positive integer in that its square is an integer of opposite sign. But the product of two charges is not a charge; there is no comparison.

Two other properties of electric charge are essential in the electrical structure of matter: Charge is *conserved,* and charge is *quantized.* These properties involve *quantity* of charge and thus imply a measurement of charge. Presently we shall state precisely how charge can be measured in terms of the force between charges a certain distance apart, and so on. But let us take this for granted for the time being, so that we may talk freely about these fundamental facts.

FIGURE 1.1
Charged particles are created in pairs with equal and opposite charge.



CONSERVATION OF CHARGE

1.2 The total charge in an isolated system never changes. By *isolated* we mean that no matter is allowed to cross the boundary of the system. We could let light pass into or out of the system, since the "particles" of light, called *photons,* carry no charge at all. Within the system charged particles may vanish or reappear, but they always do so in pairs of equal and opposite charge. For instance, a thin-walled box in a vacuum exposed to gamma rays might become the scene of a "pair-creation" event in which a high-energy photon ends its existence with the creation of an electron and a positron (Fig. 1.1). Two electrically charged particles have been newly created, but the net change in total charge, in and on the box, is zero. An event that *would* violate the law we have just stated would be the creation of a positively charged particle *without* the simultaneous creation of a negatively charged particle. Such an occurrence has never been observed.

Of course, if the electric charges of an electron and a positron were not precisely equal in magnitude, pair creation would still violate the strict law of charge conservation. That equality is a manifestation of the particle–antiparticle duality already mentioned, a universal symmetry of nature.

One thing will become clear in the course of our study of electromagnetism: Nonconservation of charge would be quite incompatible with the structure of our present electromagnetic theory. We may therefore state, either as a postulate of the theory or as an empirical law supported without exception by all observations so far, the charge conservation law:

> The total electric charge in an isolated system, that is, the algebraic sum of the positive and negative charge present at any time, never changes.

Sooner or later we must ask whether this law meets the test of relativistic invariance. We shall postpone until Chapter 5 a thorough discussion of this important question. But the answer is that it does, and not merely in the sense that the statement above holds in any given inertial frame but in the stronger sense that observers in different frames, measuring the charge, obtain the same number. In other words the total electric charge of an isolated system is a relativistically invariant number.

QUANTIZATION OF CHARGE

1.3 The electric charges we find in nature come in units of one magnitude only, equal to the amount of charge carried by a single electron. We denote the magnitude of that charge by e. (When we are paying attention to sign, we write $-e$ for the charge on the electron itself.) We have already noted that the positron carries precisely that amount of charge, as it must if charge is to be conserved when an electron and a positron annihilate, leaving nothing but light. What seems more remarkable is the apparently exact equality of the charges carried by all other charged particles—the equality, for instance, of the positive charge on the proton and the negative charge on the electron.

That particular equality is easy to test experimentally. We can see whether the net electric charge carried by a hydrogen molecule, which consists of two protons and two electrons, is zero. In an experiment carried out by J. G. King,† hydrogen gas was compressed into

†J. G. King, *Phys. Rev. Lett.* **5**:562 (1960). References to previous tests of charge equality will be found in this article and in the chapter by V. W. Hughes in "Gravitation and Relativity," H. Y. Chieu and W. F. Hoffman (eds.), W. A. Benjamin, New York, 1964, chap. 13.

a tank that was electrically insulated from its surroundings. The tank contained about 5×10^{24} molecules [approximately 17 grams (gm)] of hydrogen. The gas was then allowed to escape by means which prevented the escape of any ion—a molecule with an electron missing or an extra electron attached. If the charge on the proton differed from that on the electron by, say, one part in a billion, then each hydrogen molecule would carry a charge of $2 \times 10^{-9}e$, and the departure of the whole mass of hydrogen would alter the charge of the tank by $10^{16}e$, a gigantic effect. In fact, the experiment could have revealed a residual molecular charge as small as $2 \times 10^{-20}e$, and none was observed. This proved that the proton and the electron do not differ in magnitude of charge by more than 1 part in 10^{20}.

Perhaps the equality is really *exact* for some reason we don't yet understand. It may be connected with the possibility, suggested by recent theories, that a proton can, *very* rarely, decay into a positron and some uncharged particles. If that were to occur, even the slightest discrepancy between proton charge and positron charge would violate charge conservation. Several experiments designed to detect the decay of a proton have not yet, as this is written in 1983, registered with certainty a single decay. If and when such an event is observed, it will show that exact equality of the magnitude of the charge of the proton and the charge of the electron (the positron's antiparticle) can be regarded as a corollary of the more general law of charge conservation.

That notwithstanding, there is now overwhelming evidence that the *internal* structure of all the strongly interacting particles called *hadrons*—a class which includes the proton and the neutron—involves basic units called *quarks,* whose electric charges come in multiples of $e/3$. The proton, for example, is made with three quarks, two of charge $\frac{2}{3}e$ and one with charge $-\frac{1}{3}e$. The neutron contains one quark of charge $\frac{2}{3}e$ and two quarks with charge $-\frac{1}{3}e$.

Several experimenters have searched for single quarks, either free or attached to ordinary matter. The fractional charge of such a quark, since it cannot be neutralized by any number of electrons or protons, should betray the quark's presence. So far no fractionally charged particle has been conclusively identified. There are theoretical grounds for suspecting that the liberation of a quark from a hadron is impossible, but the question remains open at this time.

The fact of charge quantization lies outside the scope of classical electromagnetism, of course. We shall usually ignore it and act as if our point charges q could have any strength whatever. This will not get us into trouble. Still, it is worth remembering that classical theory cannot be expected to explain the structure of the elementary particles. (It is not certain that present quantum theory can either!) What holds the electron together is as mysterious as what fixes the precise value of its charge. Something more than electrical forces must be involved, for the electrostatic forces between different parts of the electron would be repulsive.

In our study of electricity and magnetism we shall treat the charged particles simply as carriers of charge, with dimensions so small that their extension and structure is for most purposes quite insignificant. In the case of the proton, for example, we know from high-energy scattering experiments that the electric charge does not extend appreciably beyond a radius of 10^{-13} cm. We recall that Rutherford's analysis of the scattering of alpha particles showed that even heavy nuclei have their electric charge distributed over a region smaller than 10^{-11} cm. For the physicist of the nineteenth century a "point charge" remained an abstract notion. Today we are on familiar terms with the atomic particles. The graininess of electricity is so conspicuous in our modern description of nature that we find a point charge less of an artificial idealization than a smoothly varying distribution of charge density. When we postulate such smooth charge distributions, we may think of them as averages over very large numbers of elementary charges, in the same way that we can define the macroscopic density of a liquid, its lumpiness on a molecular scale notwithstanding.

COULOMB'S LAW

1.4 As you probably already know, the interaction between electric charges at rest is described by Coulomb's law: Two stationary electric charges repel or attract one another with a force proportional to the product of the magnitude of the charges and inversely proportional to the square of the distance between them.

We can state this compactly in vector form:

$$\mathbf{F}_2 = k\,\frac{q_1 q_2 \hat{\mathbf{r}}_{21}}{r_{21}^2} \tag{1}$$

Here q_1 and q_2 are numbers (scalars) giving the magnitude and sign of the respective charges, $\hat{\mathbf{r}}_{21}$ is the unit vector in the direction† from charge 1 to charge 2, and $\mathbf{F}_2$ is the force acting on charge 2. Thus Eq. 1 expresses, among other things, the fact that like charges repel and unlike attract. Also, the force obeys Newton's third law; that is, $\mathbf{F}_2 = -\mathbf{F}_1$.

The unit vector $\hat{\mathbf{r}}_{21}$ shows that the force is parallel to the line joining the charges. It could not be otherwise unless space itself has some built-in directional property, for with two point charges alone in empty and isotropic space, no other direction could be singled out.

†The convention we adopt here may not seem the natural choice, but it is more consistent with the usage in some other parts of physics and we shall try to follow it throughout this book.

If the point charge itself had some internal structure, with an axis defining a direction, then it would have to be described by more than the mere scalar quantity q. It is true that some elementary particles, including the electron, do have another property, called *spin.* This gives rise to a magnetic force between two electrons in addition to their electrostatic repulsion. This magnetic force does not, in general, act in the direction of the line joining the two particles. It decreases with the inverse fourth power of the distance, and at atomic distances of 10^{-8} cm the Coulomb force is already about 10^4 times stronger than the magnetic interaction of the spins. Another magnetic force appears if our charges are moving—hence the restriction to stationary charges in our statement of Coulomb's law. We shall return to these magnetic phenomena in later chapters.

Of course we must assume, in writing Eq. 1, that both charges are well localized, each occupying a region small compared with r_{21}. Otherwise we could not even define the distance r_{21} precisely.

The value of the constant k in Eq. 1 depends on the units in which r, $\mathbf{F}$, and q are to be expressed. Usually we shall choose to measure r_{21} in cm, $\mathbf{F}$ in dynes, and charge in electrostatic units (esu). Two like charges of 1 esu each repel one another with a force of 1 dyne when they are 1 cm apart. Equation 1, with $k = 1$, is the definition of the unit of charge in CGS electrostatic units, the dyne having already been defined as the force that will impart an acceleration of one centimeter per second per second to a one-gram mass. Figure 1.2*a* is just a graphic reminder of the relation. The magnitude of *e,* the fundamental quantum of electric charge, is 4.8023×10^{-10} esu.

We want to be familiar also with the unit of charge called the *coulomb.* This is the unit for electric charge in the *Système Internationale* (SI) family of units. That system is based on the meter, kilogram, and second as units of length, mass, and time, and among its electrical units are the familiar volt, ohm, ampere, and watt.

The SI unit of force is the newton, equivalent to exactly 10^5 dynes, the force that will cause a one-kilogram mass to accelerate at one meter per second per second. The coulomb is defined by Eq. 1 with $\mathbf{F}$ in newtons, r_{21} in meters, charges q_1 and q_2 in coulombs, and $k = 8.988 \times 10^9$. A charge of 1 coulomb equals 2.998×10^9 esu. Instead of k, it is customary to introduce a constant ϵ_0, which is just $(4\pi k)^{-1}$, with which the same equation is written

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 \hat{\mathbf{r}}_{21}}{r_{21}^2} \qquad (1')$$

Refer to Fig. 1.2*b* for an example. The constant ϵ_0 will appear in several SI formulas that we'll meet in the course of our study. The exact value of ϵ_0 and the exact relation of the coulomb to the esu can be found in Appendix E. For our purposes the following approximations are quite accurate enough: $k = 9 \times 10^9$; 1 coulomb $= 3 \times 10^9$ esu.

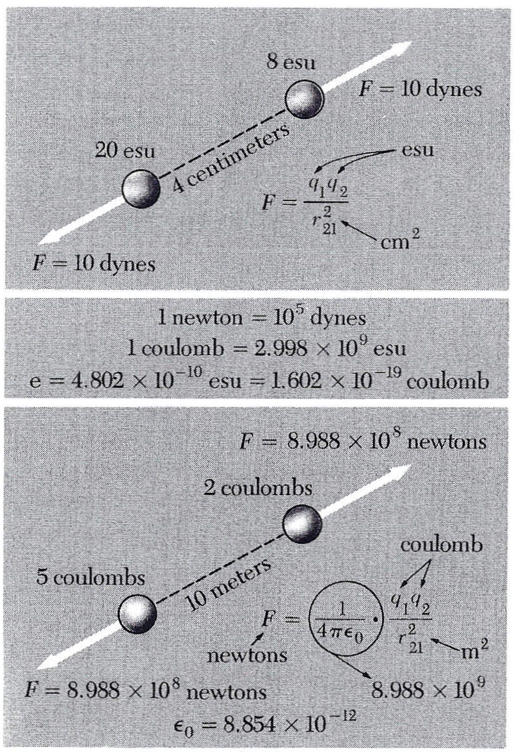


FIGURE 1.2
Coulomb's law expressed in CGS electrostatic units (top) and in SI units (bottom). The constant ϵ_0 and the factor relating coulombs to esu are connected, as we shall learn later, with the speed of light. We have rounded off the constants in the figure to four-digit accuracy. The precise values are given in Appendix E.

Fortunately the electronic charge e is very close to an easily remembered approximate value in either system: $e = 4.8 \times 10^{-10}$ esu $= 1.6 \times 10^{-19}$ coulomb.

The only way we have of detecting and measuring electric charges is by observing the interaction of charged bodies. One might wonder, then, how much of the apparent content of Coulomb's law is really only definition. As it stands, the significant physical content is the statement of inverse-square dependence and the implication that electric charge is *additive* in its effect. To bring out the latter point, we have to consider *more* than two charges. After all, if we had only two charges in the world to experiment with, q_1 and q_2, we could never measure them separately. We could verify only that F is proportional to $1/r_{21}^2$. Suppose we have *three* bodies carrying charges q_1, q_2, and q_3. We can measure the force on q_1 when q_2 is 10 cm away from q_1 and q_3 is very far away, as in Fig. 1.3*a*. Then we can take q_2 away, bring q_3 into q_2's former position, and again measure the force on q_1. Finally, we bring q_2 and q_3 very close together and locate the combination 10 cm from q_1. We find by measurement that the force on q_1 is equal to the sum of the forces previously measured. This is a significant result that could *not* have been predicted by logical arguments from symmetry like the one we used above to show that the force between two point charges *had* to be along the line joining them. *The force with which two charges interact is not changed by the presence of a third charge.*

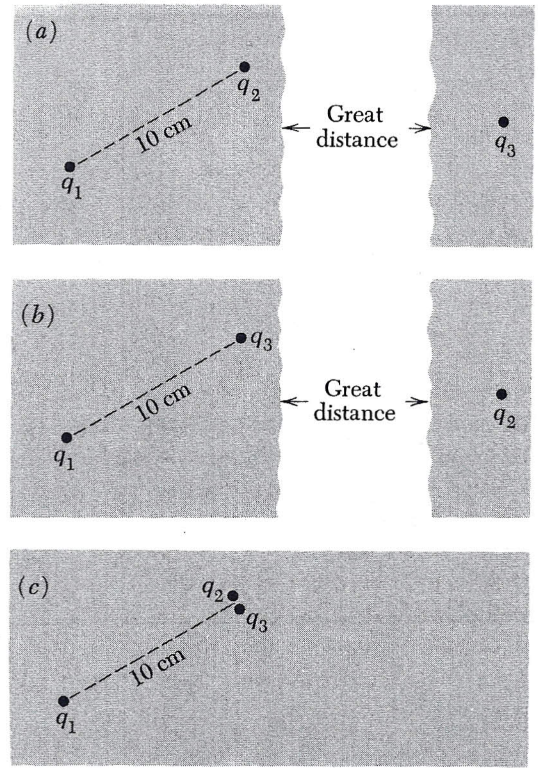


FIGURE 1.3
The force on q_1 in (*c*) is the sum of the forces on q_1 in (*a*) and (*b*).

No matter how many charges we have in our system. Coulomb's law (Eq. 1) can be used to calculate the interaction of every pair. This is the basis of the principle of *superposition,* which we shall invoke again and again in our study of electromagnetism. Superposition means combining two sets of sources into one system by adding the second system "on top of" the first without altering the configuration of either one. Our principle ensures that the force on a charge placed at any point in the combined system will be the vector sum of the forces that each set of sources, acting alone, causes to act on a charge at that point. This principle must not be taken lightly for granted. There may well be a domain of phenomena, involving very small distances or very intense forces, where superposition *no longer holds.* Indeed, we know of quantum phenomena in the electromagnetic field which do represent a failure of superposition, seen from the viewpoint of the classical theory.

Thus the physics of electrical interactions comes into full view only when we have *more* than two charges. We can go beyond the explicit statement of Eq. 1 and assert that, with the three charges in Fig. 1.3 occupying any positions whatever, the force on any one of them, such as q_3, is correctly given by this equation:

$$\mathbf{F}_3 = \frac{q_3 q_1 \hat{\mathbf{r}}_{31}}{r_{31}^2} + \frac{q_3 q_2 \hat{\mathbf{r}}_{32}}{r_{32}^2} \qquad (2)$$

The experimental verification of the inverse-square law of electrical attraction and repulsion has a curious history. Coulomb himself announced the law in 1786 after measuring with a torsion balance the force between small charged spheres. But 20 years earlier Joseph Priestly, carrying out an experiment suggested to him by Benjamin Franklin, had noticed the absence of electrical influence within a hollow charged container and made an inspired conjecture: "May we not infer from this experiment that the attraction of electricity is subject to the same laws with that of gravitation and is therefore according to the square of the distances; since it is easily demonstrated that were the earth in the form of a shell, a body in the inside of it would not be attracted to one side more than the other."† The same idea was the basis of an elegant experiment in 1772 by Henry Cavendish. Cavendish charged a spherical conducting shell which contained within it, and temporarily connected to it, a smaller sphere. The outer shell was then separated into two halves and carefully removed, the inner sphere having been first disconnected. This sphere was tested for charge, the absence of which would confirm the inverse-square law. Assuming that a deviation from the inverse-square law could be expressed as a difference in the exponent, $2 + \delta$, say, instead of 2, Cavendish concluded that δ must be less than 0.03. This experiment of Cavendish remained largely unknown until Maxwell discovered and published Cavendish's notes a century later (1876). At that time also Maxwell repeated the experiment with improved apparatus, pushing the limit down to $\delta < 10^{-6}$. The latest of several modern versions of the Cavendish experiment,‡ if interpreted the same way, yielded the fantastically small limit $\delta < 10^{-15}$.

During the second century after Cavendish, however, the question of interest changed somewhat. Never mind how perfectly Coulomb's law works for charged objects in the laboratory—is there a range of distances where it completely breaks down? There are two domains in either of which a breakdown is conceivable. The first is the domain of very small distances, distances less than 10^{-14} cm where electromagnetic theory as we know it may not work at all. As for very large distances, from the geographical, say, to the astronomical, a test of Coulomb's law by the method of Cavendish is obviously not feasible. Nevertheless we do observe certain large-scale electromagnetic phenomena which prove that the laws of classical electromagnetism work over very long distances. One of the most stringent tests is provided by planetary magnetic fields, in particular, the magnetic field of the giant planet Jupiter, which was surveyed in the mission of Pioneer

†Joseph Priestly, "The History and Present State of Electricity," vol. II, London, 1767.

‡E. R. Williams, J. G. Faller, and H. Hill. *Phys. Rev. Lett.* **26**:721 (1971).

10. The spatial variation of this field was carefully analyzed† and found to be entirely consistent with classical theory out to a distance of at least 10^5 kilometers (km) from the planet. This is tantamount to a test, albeit indirect, of Coulomb's law over that distance.

To summarize, we have every reason for confidence in Coulomb's law over the stupendous range of 24 decades in distance, from 10^{-14} to 10^{10} cm, if not farther, and we take it as the foundation of our description of electromagnetism.

ENERGY OF A SYSTEM OF CHARGES

1.5 In principle, Coulomb's law is all there is to electrostatics. Given the charges and their locations we can find all the electrical forces. Or given that the charges are free to move under the influence of other kinds of forces as well, we can find the equilibrium arrangement in which the charge distribution will remain stationary. In the same sense, Newton's laws of motion are all there is to mechanics. But in both mechanics and electromagnetism we gain power and insight by introducing other concepts, most notably that of energy.

Energy is a useful concept here because electrical forces are *conservative*. When you push charges around in electric fields, no energy is irrecoverably lost. Everything is perfectly reversible. Consider first the work which must be done *on* the system to bring some charged bodies into a particular arrangement. Let us start with two charged bodies or particles very far apart from one another, as indicated at the top of Fig. 1.4, carrying charges q_1 and q_2. Whatever energy may have been needed to create these two concentrations of charge originally we shall leave entirely out of account. Bring the particles slowly together until the distance between them is r_{12}. How much work does this take?

It makes no difference whether we bring q_1 toward q_2 or the other way around. In either case the work done is the integral of the product: force times displacement in direction of force. The force that has to be applied to move one charge toward the other is equal to and opposite the Coulomb force.

$$W = \int \text{force} \times \text{distance} = \int_{r=\infty}^{r_{12}} \frac{q_1 q_2(-dr)}{r^2} = \frac{q_1 q_2}{r_{12}} \qquad (3)$$

Because r is changing from ∞ to r_{12}, the increment of displacement is $-dr$. We know the work done on the system must be positive for charges of like sign; they have to be pushed together. With q_1 and q_2 in esu, and r_{12} in cm, Eq. 3 gives the work in ergs.

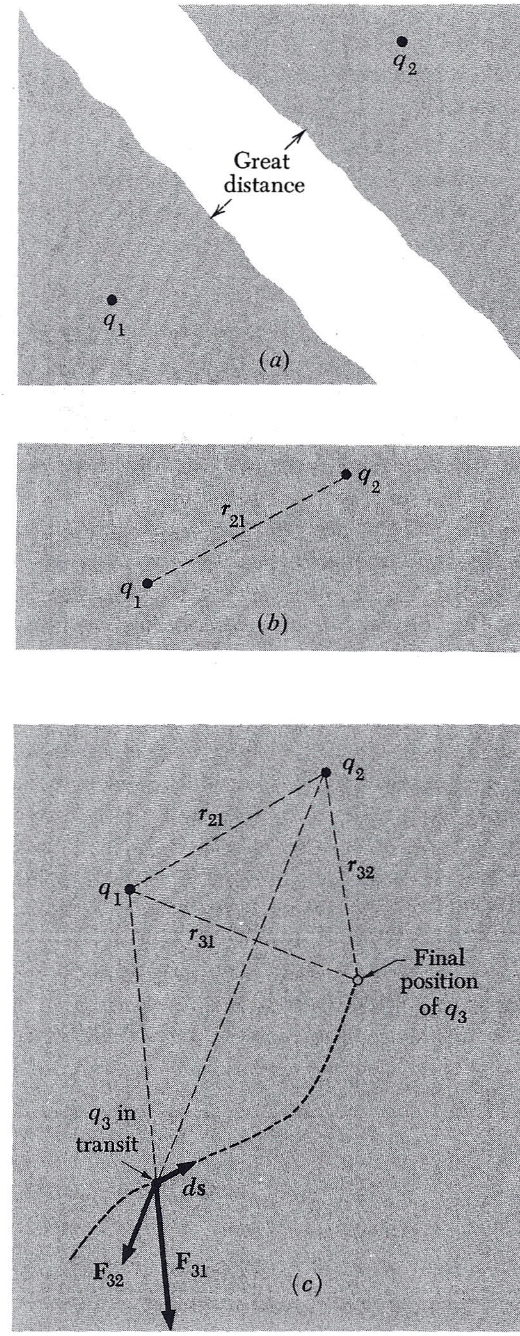


FIGURE 1.4
Three charges are brought near one another. First q_2 is brought in; then with q_1 and q_2 fixed, q_3 is brought in.

†L. Davis, Jr., A. S. Goldhaber, M. M. Nieto, *Phys. Rev. Lett.* **35**:1402 (1975). For a review of the history of the exploration of the outer limit of classical electromagnetism, see A. S. Goldhaber and M. M. Nieto, *Rev. Mod. Phys.* **43**:277 (1971).

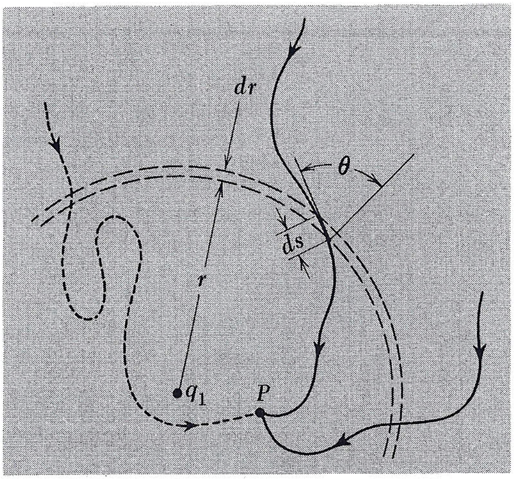


FIGURE 1.5
Because the force is central, the sections of different paths between $r + dr$ and r require the same amount of work.

This work is the same whatever the path of approach. Let's review the argument as it applies to the two charges q_1 and q_2 in Fig. 1.5. There we have kept q_1 fixed, and we show q_2 moved to the same final position along two different paths. Every spherical shell such as the one indicated between r and $r + dr$ must be crossed by both paths. The increment of work involved, $-\mathbf{F} \cdot d\mathbf{s}$ in this bit of path, is the same for the two paths.† The reason is that $\mathbf{F}$ has the same magnitude at both places and is directed radially from q_1, while $ds = dr/\cos\theta$; hence $\mathbf{F} \cdot d\mathbf{s} = F\,dr$. Each increment of work along one path is matched by a corresponding increment on the other, so the sums must be equal. Our conclusion holds even for paths that loop in and out, like the dotted path in Fig. 1.5. (Why?)

Returning now to the two charges as we left them in Fig. 1.4*b*, let us bring in from some remote place a third charge q_3 and move it to a point P_3 whose distance from charge 1 is r_{31} cm, and from charge 2, r_{32} cm. The work required to effect this will be

$$W_3 = -\int_{\infty}^{P_3} \mathbf{F}_3 \cdot d\mathbf{s} \tag{4}$$

Thanks to the additivity of electrical interactions, which we have already emphasized,

$$\begin{aligned} -\int \mathbf{F}_3 \cdot d\mathbf{s} &= -\int (\mathbf{F}_{31} + \mathbf{F}_{32}) \cdot d\mathbf{s} \\ &= -\int \mathbf{F}_{31} \cdot d\mathbf{r} - \int \mathbf{F}_{32} \cdot d\mathbf{r} \end{aligned} \tag{5}$$

That is, the work required to bring q_3 to P_3 is the sum of the work needed when q_1 is present alone and that needed when q_2 is present alone.

$$W_3 = \frac{q_1 q_3}{r_{31}} + \frac{q_2 q_3}{r_{32}} \tag{6}$$

The total work done in assembling this arrangement of three charges, which we shall call U, is therefore

$$U = \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \tag{7}$$

We note that q_1, q_2, and q_3 appear symmetrically in the expression above, in spite of the fact that q_3 was brought up last. We would have reached the same result if q_3 had been brought in first. (Try it.) Thus U is independent of the *order* in which the charges were assem-

†Here we use for the first time the scalar product, or "dot product," of two vectors. A reminder: the scalar product of two vectors $\mathbf{A}$ and $\mathbf{B}$, written $\mathbf{A} \cdot \mathbf{B}$, is the number $A\,B\cos\theta$. A and B are the magnitudes of the vectors $\mathbf{A}$ and $\mathbf{B}$, and θ is the angle between them. Expressed in terms of cartesian components of the two vectors, $\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$.

bled. Since it is independent also of the route by which each charge was brought in, U must be a unique property of the final arrangement of charges. We may call it the *electrical potential energy* of this particular system. There is a certain arbitrariness, as always, in the definition of a potential energy. In this case we have chosen the zero of potential energy to correspond to the situation with the three charges already in existence but infinitely far apart from one another. The potential energy *belongs to the configuration as a whole.* There is no meaningful way of assigning a certain fraction of it to one of the charges.

It is obvious how this very simple result can be generalized to apply to any number of charges. If we have N different charges, in any arrangement in space, the potential energy of the system is calculated by summing over all pairs, just as in Eq. 7. The zero of potential energy, as in that case, corresponds to all charges far apart.

As an example, let us calculate the potential energy of an arrangement of eight negative charges on the corners of a cube of side b, with a positive charge in the center of the cube, as in Fig. 1.6*a*. Suppose each negative charge is an electron with charge $-e$, while the central particle carries a double positive charge, $2e$. Summing over all pairs, we have

$$U = \frac{8(-2e^2)}{(\sqrt{3}/2)b} + \frac{12e^2}{b} + \frac{12e^2}{\sqrt{2}b} + \frac{4e^2}{\sqrt{3}b} = \frac{4.32e^2}{b} \qquad (8)$$

Figure 1.6*b* shows where each term in this sum comes from. The energy is positive, indicating that work had to be done on the system to assemble it. That work could, of course, be recovered if we let the charges move apart, exerting forces on some external body or bodies. Or if the electrons were simply to fly apart from this configuration, the *total kinetic energy* of all the particles would become equal to U. This would be true whether they came apart simultaneously and symmetrically, or were released one at a time in any order. Here we see the power of this simple notion of the total potential energy of the system. Think what the problem would be like if we had to compute the resultant vector force on every particle at every stage of assembly of the configuration! In this example, to be sure, the geometrical symmetry would simplify that task; even so, it would be more complicated than the simple calculation above.

One way of writing the instruction for the sum over pairs is this:

$$U = \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{q_j q_k}{r_{jk}} \qquad (9)$$

The double-sum notation, $\Sigma_{j=1}^{N}\Sigma_{k\neq j}$, says: Take $j = 1$ and sum over $k = 2, 3, 4, \ldots, N$; then take $j = 2$ and sum over $k = 1, 3, 4, \ldots, N$; and so on, through $j = N$. Clearly this includes every pair *twice,* and to correct for that we put in front the factor ½.

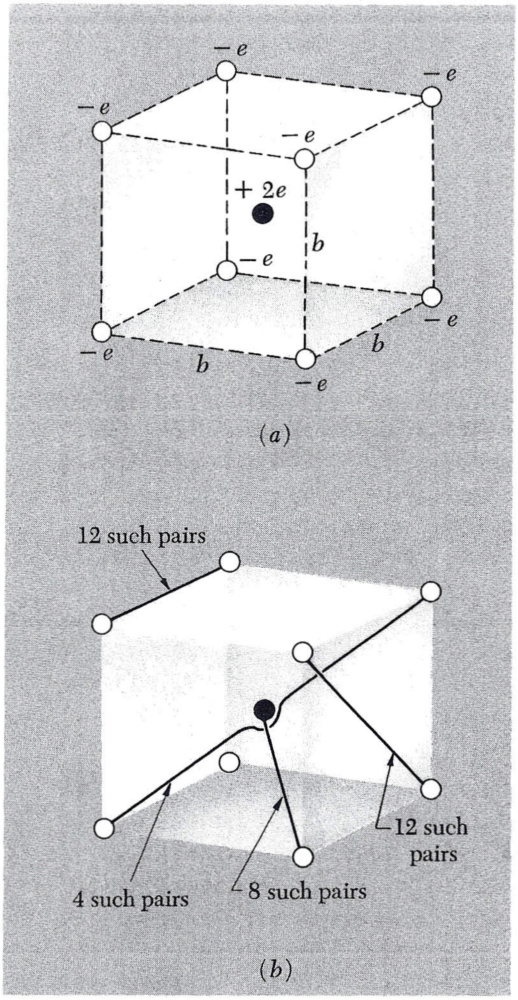


FIGURE 1.6
(*a*) The potential energy of this arrangement of nine point charges is given by Eq. 9. (*b*) Four types of pairs are involved in the sum.

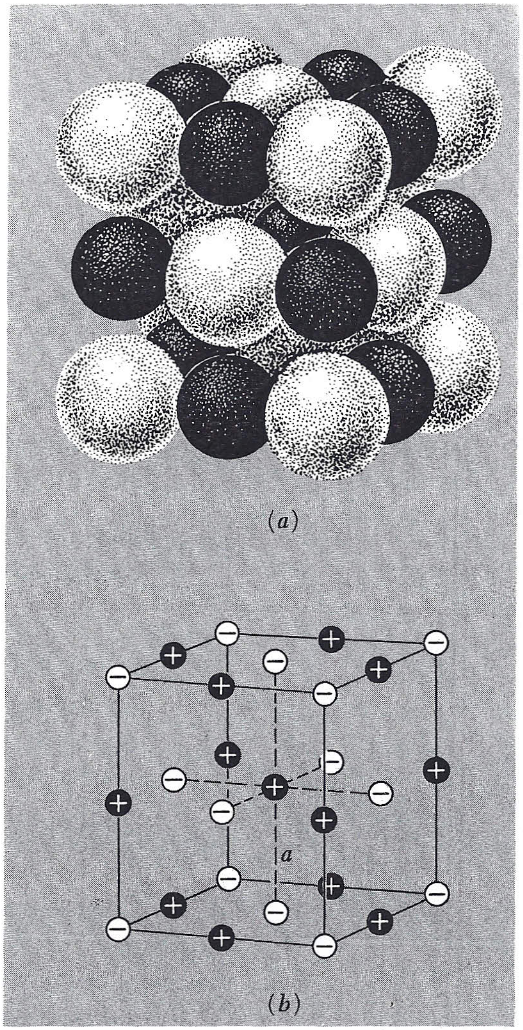


FIGURE 1.7
A portion of a sodium chloride crystal, with the ions Na^+ and $C1^-$ shown in about the right relative proportions (*a*), and replaced by equivalent point charges (*b*).

ELECTRICAL ENERGY IN A CRYSTAL LATTICE

1.6 These ideas have an important application in the physics of crystals. We know that an ionic crystal like sodium chloride can be described, to a very good approximation, as an arrangement of positive ions (Na^+) and negative ions (Cl^-) alternating in a regular three-dimensional array or lattice. In sodium chloride the arrangement is that shown in Fig. 1.7*a*. Of course the ions are not point charges, but they are nearly spherical distributions of charge and therefore (as we shall presently prove) the electrical forces they exert on one another are the same as if each ion were replaced by an equivalent point charge at its center. We show this electrically equivalent system in Fig. 1.7*b*. The electrostatic potential energy of the lattice of charges plays an important role in the explanation of the stability and cohesion of the ionic crystal. Let us see if we can estimate its magnitude.

We seem to be faced at once with a sum that is enormous, if not doubly infinite, for any macroscopic crystal contains 10^{20} atoms at least. Will the sum converge? Now what we hope to find is the potential energy per unit volume or mass of crystal. We confidently expect this to be independent of the size of the crystal, based on the general argument that one end of a macroscopic crystal can have little influence on the other. Two grams of sodium chloride ought to have twice the potential energy of 1 gm, and the shape should not be important so long as the surface atoms are a small fraction of the total number of atoms. We would be *wrong* in this expectation if the crystal were made out of ions of one sign only. Then, 1 gm of crystal would carry an enormous electric charge, and putting two such crystals together to make a 2-gm crystal would take a fantastic amount of energy. (You might estimate how much!) The situation is saved by the fact that the crystal structure is an alternation of equal and opposite charges, so that any macroscopic bit of crystal is very nearly neutral.

To evaluate the potential energy we first observe that every positive ion is in a position equivalent to that of every other positive ion. Furthermore, although it is perhaps not immediately obvious from Fig. 1.7, the arrangement of positive ions around a negative ion is exactly the same as the arrangement of negative ions around a positive ion, and so on. Hence we may take one ion as a center, it matters not which kind, sum over *its* interactions with all the others, and simply multiply by the total number of ions of both kinds. This reduces the double sum in Eq. 9, to a single sum and a factor N; we must still apply the factor ½ to compensate for including each pair twice. That is, the energy of a sodium chloride lattice composed of a total of N ions is

$$U = \frac{1}{2} N \sum_{k=2}^{N} \frac{q_1 q_k}{r_{1k}} \tag{10}$$

Taking the positive ion at the center as in Fig. 1.7*b*, our sum runs over all its neighbors near and far. The leading terms start out as follows:

$$U = \frac{1}{2}N\left(-\frac{6e^2}{a} + \frac{12e^2}{\sqrt{2}a} - \frac{8e^2}{\sqrt{3}a} + \cdots\right) \tag{11}$$

The first term comes from the 6 nearest chlorine ions, at distance a, the second from the 12 sodium ions on the cube edges, and so on. It is clear, incidentally, that this series does not converge *absolutely;* if we were so foolish as to try to sum all the positive terms first, that sum would diverge. To evaluate such a sum, we should arrange it so that as we proceed outward, including ever more distant ions, we include them in groups which represent nearly neutral shells of material. Then if the sum is broken off, the more remote ions which have been neglected will be such an even mixture of positive and negative charges that we can be confident their contribution would have been small. This is a crude way to describe what is actually a somewhat more delicate computational problem. The numerical evaluation of such a series is easily accomplished with a computer. The answer in this example happens to be

$$U = \frac{-0.8738Ne^2}{a} \tag{12}$$

Here N, the number of ions, is twice the number of NaCl molecules.

The negative sign shows that work would have to be *done* to take the crystal apart into ions. In other words, the electrical energy helps to explain the cohesion of the crystal. If this were the whole story, however, the crystal would collapse, for the potential energy of the charge distribution is obviously *lowered* by shrinking all the distances. We meet here again the familiar dilemma of classical—that is, non-quantum—physics. No system of stationary particles can be in stable equilibrium, according to classical laws, under the action of electrical forces alone. Does this make our analysis useless? Not at all. Remarkably, and happily, in the quantum physics of crystals the electrical potential energy can still be given meaning, and can be computed very much in the way we have learned here.

THE ELECTRIC FIELD

1.7 Suppose we have some arrangement of charges, $q_1, q_2, \ldots, q_N$, fixed in space, and we are interested not in the forces they exert on one another but only in their effect on some other charge q_0 which might be brought into their vicinity. We know how to calculate the

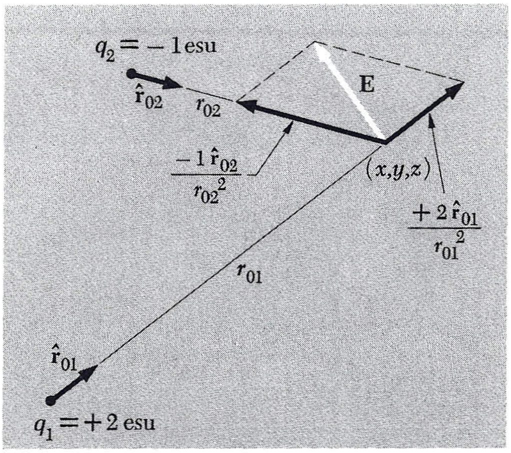


FIGURE 1.8
The field at a point is the vector sum of the fields of each of the charges in the system.

resultant force on this charge, given its position which we may specify by the coordinates x, y, z. The force on the charge q_0 is

$$\mathbf{F}_0 = \sum_{j=1}^{N} \frac{q_0 q_j \hat{\mathbf{r}}_{0j}}{r_{0j}^2} \tag{13}$$

where $\mathbf{r}_{0j}$ is the vector from the jth charge in the system to the point (x, y, z). The force is proportional to q_0, so if we divide out q_0 we obtain a vector quantity which depends only on the structure of our original system of charges, $q_1, \ldots, q_N$, and on the position of the point (x, y, z). We call this vector function of x, y, z the *electric field* arising from the $q_1, \ldots, q_N$ and use the symbol $\mathbf{E}$ for it. The charges $q_1, \ldots, q_N$ we call *sources* of the field. We may take as the *definition* of the electric field $\mathbf{E}$ of a charge distribution, at the point (x, y, z)

$$\mathbf{E}(x, y, z) = \sum_{j=1}^{N} \frac{q_j \hat{\mathbf{r}}_{0j}}{r_{0j}^2} \tag{14}$$

Figure 1.8 illustrates the vector addition of the field of a point charge of 2 esu to the field of a point charge of -1 esu, at a particular point in space. In the CGS system of units, electric field strength is expressed in dynes per unit charge, that is, dynes/esu.

In SI units with the coulomb as the unit of charge and the newton as the unit of force, the electric field strength $\mathbf{E}$ can be expressed in newtons/coulomb, and Eq. 14 would be written like this:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{N} \frac{q_j \hat{\mathbf{r}}_{0j}}{r_{0j}^2} \tag{14$'$}$$

each distance r_{0j} being measured in meters.

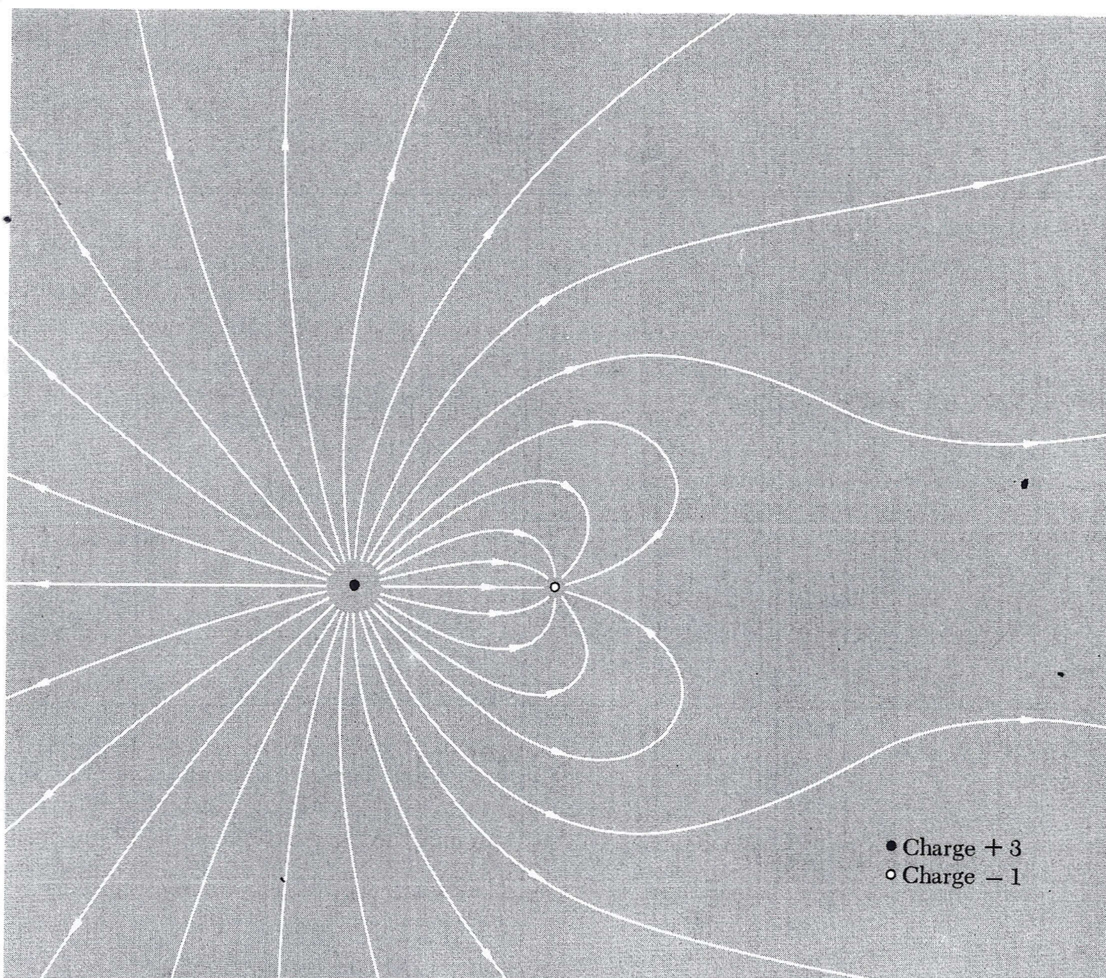
After the introduction of the electric potential in the next chapter, we shall have another, and completely equivalent, way of expressing the unit of electric field strength; namely, statvolts/cm in the CGS system of units and volts/meter in SI units.

So far we have nothing really new. The electric field is merely another way of describing the system of charges; it does so by giving the force per unit charge, in magnitude and direction, that an exploring charge q_0 would experience at any point. We have to be a little careful with that interpretation. Unless the source charges are really immovable, the introduction of some finite charge q_0 may cause the source charges to shift their positions, so that the field itself, as defined by Eq. 14, is different. That is why we assumed fixed charges to begin our discussion. People sometimes define the field by requiring q_0 to be an "infinitesimal" test charge, letting $\mathbf{E}$ be the limit of $\mathbf{F}/q_0$ as $q_0 \to 0$. Any flavor of rigor this may impart is illusory. Remember that in the real world we have never observed a charge smaller than e! Actually, if we take Eq. 14 as our *definition* of $\mathbf{E}$, without reference to a test charge, no problem arises and the sources need not be fixed.

ure rotated around the symmetry axis. In Fig. 1.10 there is one point in space where **E** is zero. How far from the nearest charge must this point lie? Notice also that toward the edge of the picture the field points more or less radially outward all around. One can see that at a very large distance from the charges the field will look very much like the field from a positive point charge. This is to be expected because the separation of the charges cannot make very much difference for points far away, and a point charge of 2 units is just what we would have left if we superimposed our two sources at one spot.

Another way to depict a vector field is to draw *field lines*. These are simply curves whose tangent, at any point, lies in the direction of the field at that point. Such curves will be smooth and continuous

FIGURE 1.11
Some field lines in the electric field around two charges, $q_1 = +3$, $q_2 = -1$.



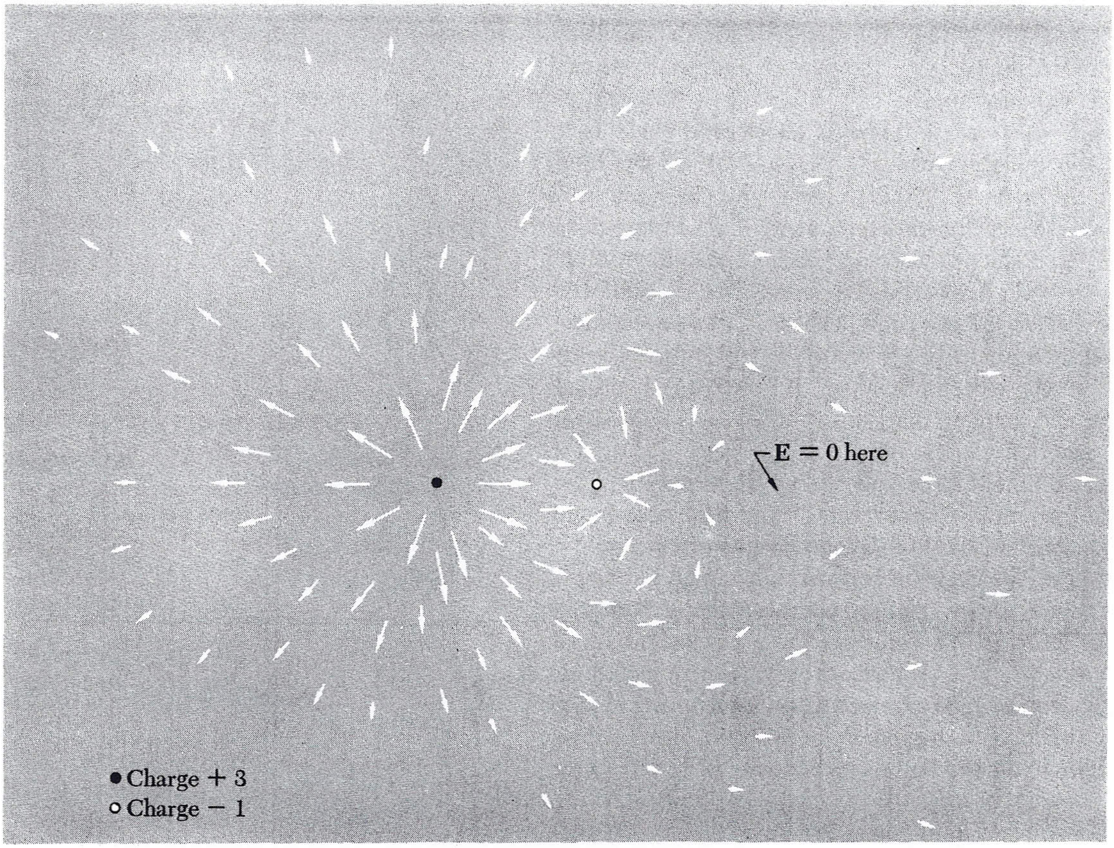


FIGURE 1.10
The field in the vicinity of two charges, $q_1 = +3$, $q_2 = -1$, is the superposition of the fields in Fig. 1.9*a* and *b*.

tion in three-dimensional space. We can indicate the magnitude and direction of **E** at various points by drawing little arrows near those points, making the arrows longer where E is larger.† Using this scheme, we show in Fig. 1.9*a* the field of an isolated point charge of 3 units and in Fig. 1.9*b* the field of a point charge of -1 unit. These pictures admittedly add nothing whatever to our understanding of the field of an isolated charge; anyone can imagine a simple radial inverse-square field without the help of a picture. We show them in order to combine the two fields in Fig. 1.10, which indicates in the same manner the field of two such charges separated by a distance a. All that Fig. 1.10 can show is the field in a plane containing the charges. To get a full three-dimensional representation one must imagine the fig-

†Such a representation is rather clumsy at best. It is hard to indicate the point in space to which a particular vector applies, and the range of magnitudes of E is usually so large that it is impracticable to make the lengths of the arrows proportional to E.

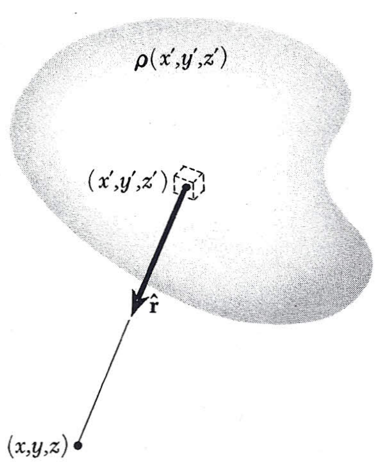
except at singularities such as point charges, or points like the one in the example of Fig. 1.10 where the field is zero. A field line plot does not directly give the magnitude of the field, although we shall see that, in a general way, the field lines converge as we approach a region of strong field and spread apart as we approach a region of weak field. In Fig. 1.11 are drawn some field lines for the same arrangement of charges as in Fig. 1.10, a positive charge of 3 units and a negative charge of 1 unit. Again, we are restricted by the nature of paper and ink to a two-dimensional section through a three-dimensional bundle of curves.

CHARGE DISTRIBUTIONS

1.8 This is as good a place as any to generalize from *point charges* to *continuous charge distributions.* A volume distribution of charge is described by a scalar charge-density function ρ, which is a function of position, with the dimensions *charge/volume*. That is, ρ times a volume element gives the amount of charge contained in that volume element. The same symbol is often used for mass per unit volume, but in this book we shall always give charge per unit volume first call on the symbol ρ. If we write ρ as a function of the coordinates x, y, z, then $\rho(x, y, z)\, dx\, dy\, dz$ is the charge contained in the little box, of volume $dx\, dy\, dz$, located at the point (x, y, z).

On an atomic scale, of course, the charge density varies enormously from point to point; even so, it proves to be a useful concept in that domain. However, we shall use it mainly when we are dealing with large-scale systems, so large that a volume element $dv = dx\, dy\, dz$ can be quite small relative to the size of our system, although still large enough to contain many atoms or elementary charges. As we have remarked before, we face a similar problem in defining the ordinary mass density of a substance.

FIGURE 1.12
Each element of the charge distribution $\rho(x', y', z')$ makes a contribution to the electric field **E** at this point (x, y, z). The total field at this point is the sum of all such contributions (Eq. 15).



If the source of the electric field is to be a continuous charge distribution rather than point charges, we merely replace the sum in Eq. 14 with the appropriate integral. The integral gives the electric field at (x, y, z), which is produced by charges at other points (x', y', z').

$$\mathbf{E}(x, y, z) = \int \frac{\rho(x', y', z')\,\hat{\mathbf{r}}\, dx'\, dy'\, dz'}{r^2} \tag{15}$$

This is a volume integral. Holding (x, y, z) fixed we let the variables of integration, x', y', and z', range over all space containing charge, thus summing up the contributions of all the bits of charge. The unit vector $\hat{\mathbf{r}}$ points from (x', y', z') to (x, y, z)—unless we want to put a minus sign before the integral, in which case we may reverse the direction of $\hat{\mathbf{r}}$. It is always hard to keep signs straight. Let's remember that the electric field points *away* from a positive source (Fig. 1.12).

In the neighborhood of a true point charge the electric field grows infinite like $1/r^2$ as we approach the point. It makes no sense to talk about the field *at* the point charge. As our ultimate physical sources of field are not, we believe, infinite concentrations of charge in zero volume but instead finite structures, we simply ignore the mathematical singularities implied by our point-charge language and rule out of bounds the interior of our elementary sources. A continuous charge distribution $\rho(x', y', z')$ which is nowhere infinite gives no trouble at all. Equation 15 can be used to find the field at any point within the distribution. The integrand doesn't blow up at $r = 0$ because the volume element in the numerator is in that limit proportional to $r^2\, dr$. That is to say, so long as ρ remains finite, the field will remain finite everywhere, even in the interior or on the boundary of a charge distribution.

FLUX

1.9 The relation between the electric field and its sources can be expressed in a remarkably simple way, one that we shall find very useful. For this we need to define a quantity called *flux*.

Consider some electric field in space and in this space some arbitrary closed surface, like a balloon of any shape. Figure 1.13 shows such a surface, the field being suggested by a few field lines. Now divide the whole surface into little patches which are so small that over any one patch the surface is practically flat and the vector field does not change appreciably from one part of a patch to another. In other words, don't let the balloon be too crinkly, and don't let its surface pass right through a singularity† of the field such as a point charge. The area of a patch has a certain magnitude in cm^2, and a patch defines a unique direction—the outward-pointing normal to its surface. (Since the surface is closed, you can tell its inside from its outside; there is no ambiguity.) Let this magnitude and direction be represented by a vector. Then for every patch into which the surface has been divided, such as patch number j, we have a vector $\mathbf{a}_j$ giving its area and orientation. The steps we have just taken are pictured in Fig. 1.13*b* and *c*. Note that the vector $\mathbf{a}_j$ does not depend at all on the shape of the patch; it doesn't matter how we have divided up the surface, as long as the patches are small enough.

Let $\mathbf{E}_j$ be the electric field vector at the location of patch number j. The scalar product $\mathbf{E}_j \cdot \mathbf{a}_j$ is a number. We call this number the *flux* through that bit of surface. To understand the origin of the name,

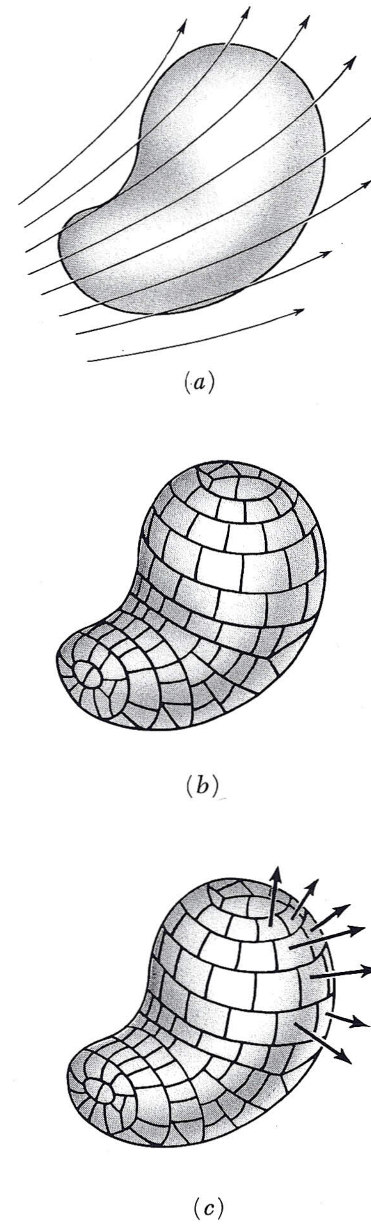


FIGURE 1.13
(*a*) A closed surface in a vector field is divided (*b*) into small elements of area. (*c*) Each element of area is represented by an outward vector.

†By a singularity of the field we would ordinarily mean not only a point source where the field approaches infinity, but any place where the field changes magnitude or direction discontinuously, such as an infinitesimally thin layer of concentrated charge. Actually this latter, milder, kind of singularity would cause no difficulty here unless our balloon's surface were to coincide with the surface of discontinuity over some finite area.

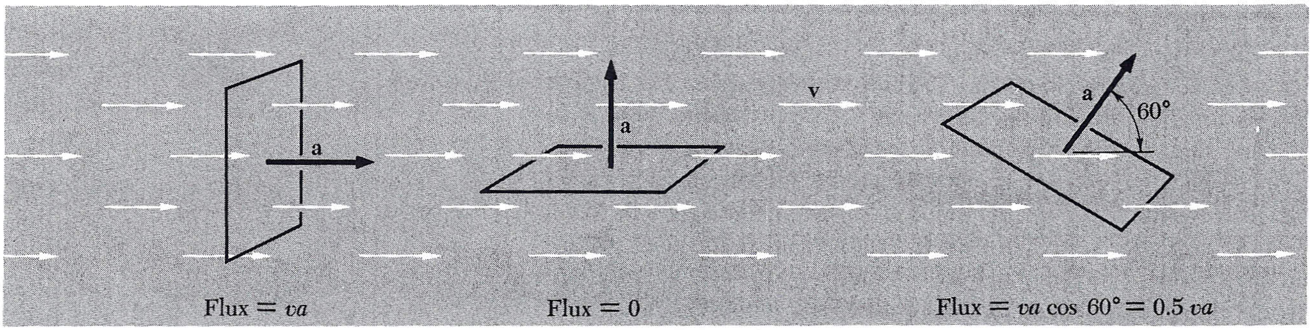


FIGURE 1.14
The flux through the frame of area **a** is **v** · **a**, where **v** is the velocity of the fluid. The flux is the volume of fluid passing through the frame, per unit time.

imagine a vector function which represents the velocity of motion in a fluid—say in a river, where the velocity varies from one place to another but is constant in time at any one position. Denote this vector field by **v**, measured, say, in meters/sec. Then, if **a** is the oriented area in square meters of a frame lowered into the water, $\mathbf{v} \cdot \mathbf{a}$ is the *rate of flow* of water through the frame in cubic meters per second (Fig. 1.14). We must emphasize that our definition of flux is applicable to any vector function, whatever physical variable it may represent.

Now let us add up the flux through all the patches to get the flux through the entire surface, a scalar quantity which we shall denote by Φ:

$$\Phi = \sum_{\text{All } j} \mathbf{E}_j \cdot \mathbf{a}_j \tag{16}$$

Letting the patches become smaller and more numerous without limit, we pass from the sum in Eq. 16 to a surface integral:

$$\Phi = \int_{\substack{\text{Entire}\\ \text{surface}}} \mathbf{E} \cdot d\mathbf{a} \tag{17}$$

A surface integral of any vector function **F**, over a surface S, means just this: Divide S into small patches, each represented by a vector outward, of magnitude equal to the patch area; at every patch, take the scalar product of the patch area vector and the local **F**; sum all these products, and the limit of this sum, as the patches shrink, is the surface integral. Do not be alarmed by the prospect of having to perform such a calculation for an awkwardly shaped surface like the one in Fig. 1.13. The surprising property we are about to demonstrate makes that unnecessary!

GAUSS'S LAW

1.10 Take the simplest case imaginable; suppose the field is that of a single isolated positive point charge q and the surface is a sphere of radius r centered on the point charge (Fig. 1.15). What is the flux Φ through this surface? The answer is easy because the magnitude of **E** at every point on the surface is q/r^2 and its direction is the same as that of the outward normal at that point. So we have

$$\Phi = E \times \text{total area} = \frac{q}{r^2} \times 4\pi r^2 = 4\pi q \qquad (18)$$

The flux is independent of the size of the sphere.

Now imagine a second surface, or balloon, enclosing the first, but *not* spherical, as in Fig. 1.16. We claim that the total flux through this surface is the same as that through the sphere. To see this, look at a cone, radiating from q, which cuts a small patch **a** out of the sphere and continues on to the outer surface where it cuts out a patch **A** at a distance R from the point charge. The area of the patch **A** is larger than that of the patch **a** by two factors: first, by the ratio of the distance squared $(R/r)^2$; and second, owing to its inclination, by the factor $1/\cos\theta$. The angle θ is the angle between the outward normal and the radial direction (see Fig. 1.16). The electric field in that neighborhood is reduced from its magnitude on the sphere by the factor $(r/R)^2$ and is still radially directed. Letting $\mathbf{E}_{(R)}$ be the field at the outer patch and $\mathbf{E}_{(r)}$ be the field at the sphere, we have

$$\text{Flux through outer patch} = \mathbf{E}_{(R)} \cdot \mathbf{A} = E_{(R)} A \cos\theta \qquad (19)$$

$$\text{Flux through inner patch} = \mathbf{E}_{(r)} \cdot \mathbf{a} = E_{(r)} a$$

$$E_{(R)} A \cos\theta = \left[E_{(r)}\left(\frac{r}{R}\right)^2\right]\left[a\left(\frac{R}{r}\right)^2 \frac{1}{\cos\theta}\right]\cos\theta = E_{(r)} a$$

This proves that the flux through the two patches is the same.

Now every patch on the outer surface can in this way be put into correspondence with part of the spherical surface, so the total flux must be the same through the two surfaces. That is, the flux through the new surface must be just $4\pi q$. But this was a surface of *arbitrary* shape and size.† We conclude: The flux of the electric field through *any* surface enclosing a point charge q is $4\pi q$. As a corollary we can say that the total flux through a closed surface is *zero* if the charge lies *outside* the surface. We leave the proof of this to the reader, along with Fig. 1.17 as a hint of one possible line of argument.

There is a way of looking at all this which makes the result seem obvious. Imagine at q a source which emits particles—such as bullets or photons—in all directions at a steady rate. Clearly the flux of particles through a window of unit area will fall off with the inverse square of the window's distance from q. Hence we can draw an analogy between the electric field strength E and the intensity of particle

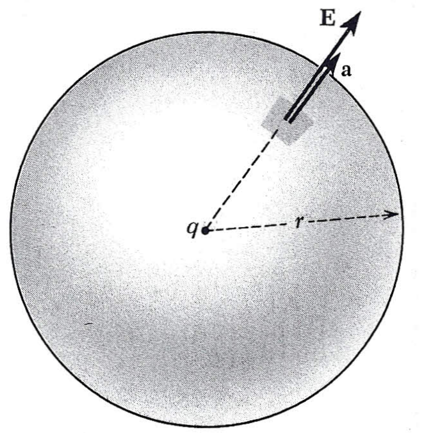


FIGURE 1.15
In the field **E** of a point charge q, what is the outward flux over a sphere surrounding q?

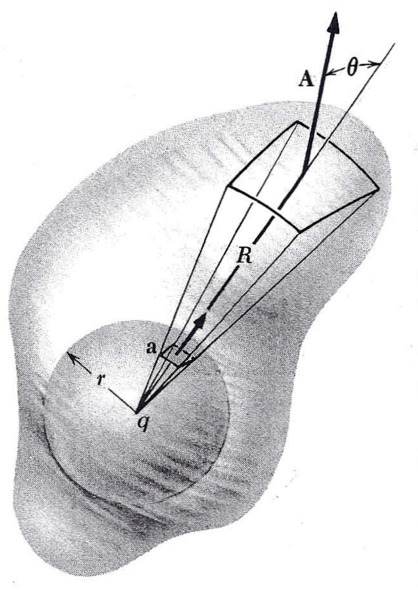


FIGURE 1.16
Showing that the flux through any closed surface around q is the same as the flux through the sphere.

†To be sure, we had the second surface enclosing the sphere, but it didn't have to, really. Besides, the sphere can be taken as small as we please.

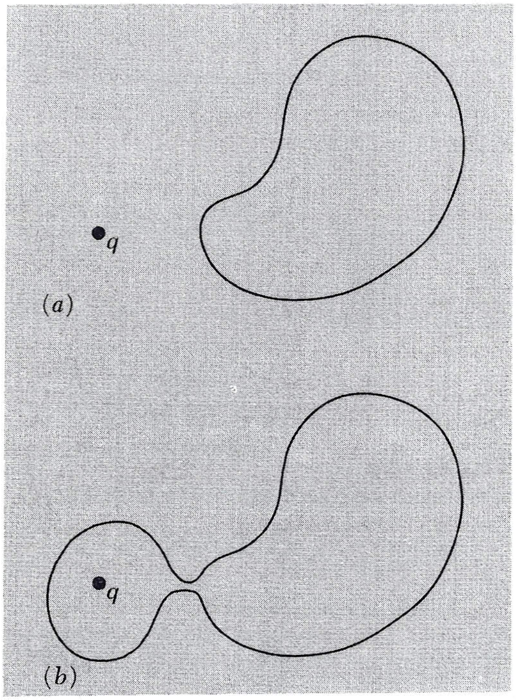


FIGURE 1.17
To show that the flux through the closed surface in (*a*) is zero, you can make use of (*b*).

flow in bullets per unit area per unit time. It is pretty obvious that the flux of bullets through any surface completely surrouding q is independent of the size and shape of that surface, for it is just the total number emitted per unit time. Correspondingly, the flux of E through the closed surface must be independent of size and shape. The common feature responsible for this is the inverse-square behavior of the intensity.

The situation is now ripe for superposition! Any electric field is the sum of the fields of its individual sources. This property was expressed in our statement, Eq. 13, of Coulomb's law. Clearly flux is an additive quantity in the same sense, for if we have a number of sources, $q_1, q_2, \ldots, q_N$, the fields of which, if each were present alone, would be $\mathbf{E}_1, \mathbf{E}_2 \ldots, \mathbf{E}_N$, the flux Φ through some surface S in the actual field can be written

$$\Phi = \int_S \mathbf{E} \cdot d\mathbf{a} = \int_S (\mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_N) \cdot d\mathbf{a} \qquad (20)$$

We have just learned that $\int_S \mathbf{E}_n \cdot d\mathbf{a}$ equals $4\pi q_n$ if the charge q_n is inside S and equals zero otherwise. So every charge q inside the surface contributes exactly $4\pi q$ to the surface integral of Eq. 20 and all charges outside contribute nothing. We have arrived at Gauss's law:

> The flux of the electric field **E** through any closed surface, that is, the integral $\int \mathbf{E} \cdot d\mathbf{a}$ over the surface, equals 4π times the total charge enclosed by the surface:
>
> $$\int \mathbf{E} \cdot d\mathbf{a} = 4\pi \sum_i q_i = 4\pi \int \rho \, dv \qquad (21)$$

We call the statement in the box a *law* because it is equivalent to Coulomb's law and it could serve equally well as the basic law of electrostatic interactions, after charge and field have been defined. Gauss's law and Coulomb's law are not two independent physical laws, but the same law expressed in different ways.†

†There is one difference, inconsequential here, but relevant to our later study of the fields of moving charges. Gauss' law is obeyed by a wider class of fields than those represented by the electrostatic field. In particular, a field that is inverse-square in r but not spherically symmetrical can satisfy Gauss' law. In other words, Gauss' law alone does not imply the symmetry of the field of a point source which is implicit in Coulomb's law.

Looking back over our proof, we see that it hinged on the inverse-square nature of the interaction and of course on the additivity of interactions, or superposition. Thus the theorem is applicable to any inverse-square field in physics, for instance, to the gravitational field.

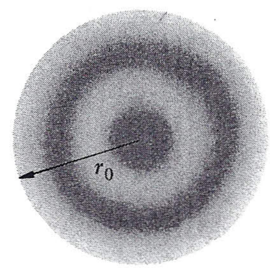
It is easy to see that Gauss's law would *not* hold if the law of force were, say, inverse-cube. For in that case the flux of electric field from a point charge q through a sphere of radius R centered on the charge would be

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{R^3} \cdot 4\pi R^2 = \frac{4\pi q}{R} \qquad (22)$$

By making the sphere large enough we could make the flux through it as small as we pleased, while the total charge inside remained constant.

This remarkable theorem enlarges our grasp in two ways. First, it reveals a connection between the field and its sources that is the converse of Coulomb's law. Coulomb's law tells us how to derive the electric field if the charges are given; with Gauss's law we can determine how much charge is in any region if the field is known. Second, the mathematical relation here demonstrated is a powerful analytic tool; it can make complicated problems easy, as we shall see.

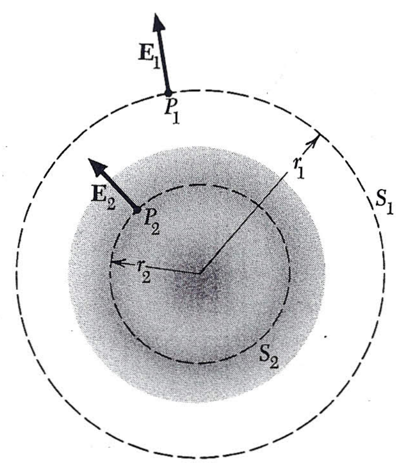
FIGURE 1.18
A charge distribution with spherical symmetry.



FIELD OF A SPHERICAL CHARGE DISTRIBUTION

1.11 We can use Gauss's law to find the electric field of a spherically symmetrical distribution of charge, that is, a distribution in which the charge density ρ depends only on the radius from a central point. Figure 1.18 depicts a cross section through some such distribution. Here the charge density is high at the center, and is zero beyond r_0. What is the electric field at some point such as P_1 outside the distribution, or P_2 inside it (Fig. 1.19)? If we could proceed only from Coulomb's law, we should have to carry out an integration which would sum the electric field vectors at P_1 arising from each elementary volume in the charge distribution. Let's try a different approach which exploits both the symmetry of the system and Gauss's law.

FIGURE 1.19
The electric field of a spherical charge distribution.



Because of the spherical symmetry, the electric field at any point must be radially directed—no other direction is unique. Likewise, the field magnitude E must be the same at all points on a spherical surface S_1 of radius r_1, for all such points are equivalent. Call this field magnitude E_1. The flux through this surface S_1 is therefore simply $4\pi r_1^2 E_1$, and by Gauss's law this must be equal to 4π times the charge enclosed by the surface. That is, $4\pi r_1^2 E_1 = 4\pi$ (charge inside S_1) or

$$E_1 = \frac{\text{charge inside } S_1}{r_1^2} \qquad (23)$$

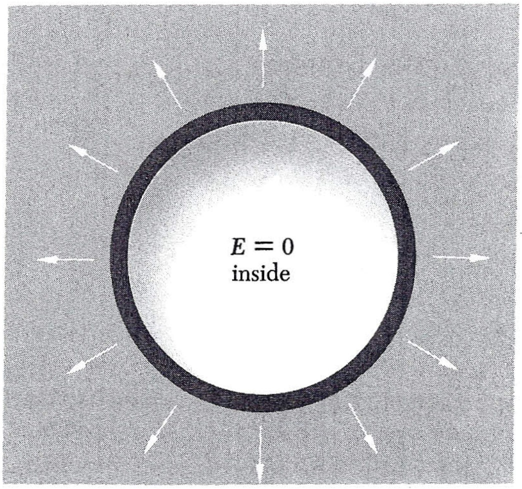


FIGURE 1.20
The field is zero inside a spherical shell of charge.

Comparing this with the field of a point charge, we see that *the field at all points on S_1 is the same as if all the charge within S_1 were concentrated at the center.* The same statement applies to a sphere drawn *inside* the charge distribution. The field at any point on S_2 is the same as if all charge within S_2 were at the center, and all charge *outside* S_2 absent. Evidently the field inside a "hollow" spherical charge distribution is zero (Fig. 1.20).

The same argument applied to the gravitational field would tell us that the earth, assuming it is spherically symmetrical in its mass distribution, attracts outside bodies as if its mass were concentrated at the center. That is a rather familiar statement. Anyone who is inclined to think the principle expresses an obvious property of the center of mass must be reminded that the theorem is not even true, in general, for other shapes. A perfect cube of uniform density does *not* attract external bodies as if its mass were concentrated at its geometrical center.

Newton didn't consider the theorem obvious. He needed it as the keystone of his demonstration that the moon in its orbit around the earth and a falling body on the earth are responding to similar forces. The delay of nearly 20 years in the publication of Newton's theory of gravitation was apparently due, in part at least, to the trouble he had in proving this theorem to his satisfaction. The proof he eventually devised and published in the *Principia* in 1686 (Book I, Section XII, Theorem XXXI) is a marvel of ingenuity in which, roughly speaking, a tricky volume integration is effected without the aid of the integral calculus as we know it. The proof is a good bit longer than our whole preceding discussion of Gauss's law, and more intricately reasoned. You see, with all his mathematical resourcefulness and originality, Newton lacked Gauss's theorem—a relation which, once it has been shown to us, seems so obvious as to be almost trivial.

FIELD OF A LINE CHARGE

1.12 A long, straight, charged wire, if we neglect its thickness, can be characterized by the amount of charge it carries per unit length. Let λ, measured in esu/cm, denote this *linear charge density*. What is the electric field of such a line charge, assumed infinitely long and with constant linear charge density λ? We'll do the problem in two ways, first by an integration starting from Coulomb's law.

To evaluate the field at the point P, shown in Fig. 1.21, we must add up the contributions from all segments of the line charge, one of which is indicated as a segment of length dx. The charge dq on this element is given by $dq = \lambda\, dx$. Having oriented our x axis along the line charge, we may as well let the y axis pass through P, which is r cm from the nearest point on the line. It is a good idea to take advantage of symmetry at the outset. Obviously the electric field at P must

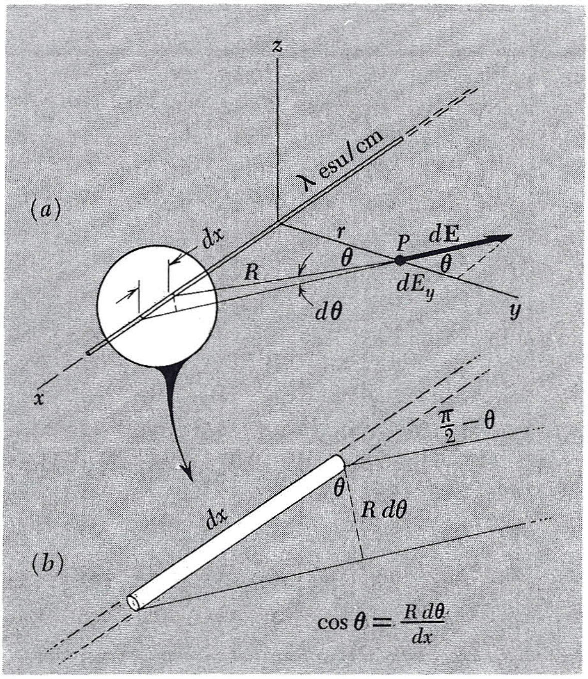


FIGURE 1.21
(*a*) The field at *P* is the vector sum of contributions from each element of the line charge. (*b*) Detail of (*a*).

point in the y direction, so that E_x and E_z are both zero. The contribution of the charge dq to the y component of the electric field at P is

$$dE_y = \frac{dq}{R^2}\cos\theta = \frac{\lambda\,dx}{R^2}\cos\theta \tag{24}$$

where θ is the angle the vector field of dq makes with the y direction. The total y component is then

$$E_y = \int dE_y = \int_{-\infty}^{\infty}\frac{\lambda\cos\theta}{R^2}\,dx \tag{25}$$

It is convenient to use θ as the variable of integration. Since $R = r/\cos\theta$ and $dx = R\,d\theta/\cos\theta$, the integral becomes

$$E_y = \int_{-\pi/2}^{\pi/2}\frac{\lambda\cos\theta\,d\theta}{r} = \frac{\lambda}{r}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = \frac{2\lambda}{r} \tag{26}$$

We see that the field of an infinitely long, uniformly dense line charge is proportional to the reciprocal of the distance from the line. Its direction is of course radially outward if the line carries a positive charge, inward if negative.

Gauss' law leads directly to the same result. Surround a segment

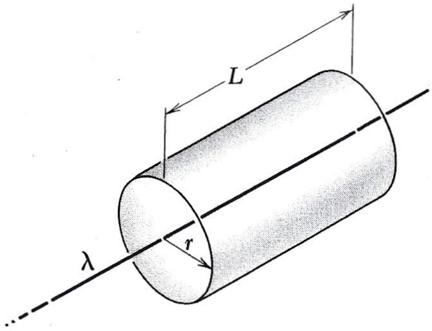


FIGURE 1.22
Using Gauss' law to find the field of a line charge.

of the line charge with a closed circular cylinder of length L and radius r, as in Fig. 1.22, and consider the flux through this surface. As we have already noted, symmetry guarantees that the field is radial, so the flux through the ends of the "tin can" is zero. The flux through the cylindrical surface is simply the area, $2\pi rL$, times E_r, the field at the surface. On the other hand, the charge enclosed by the surface is just λL, so Gauss's law gives us $2\pi rLE_r = 4\pi\lambda L$ or

$$E_r = \frac{2\lambda}{r} \tag{27}$$

in agreement with Eq. 26.

FIELD OF AN INFINITE FLAT SHEET OF CHARGE

1.13 Electric charge distributed smoothly in a thin sheet is called a *surface charge distribution.* Consider a flat sheet infinite in extent, with the constant surface charge density σ. The electric field on either side of the sheet, whatever its magnitude may turn out to be, must surely point perpendicular to the plane of the sheet; there is no other unique direction in the system. Also because of symmetry, the field must have the same magnitude and the opposite direction at two points P and P' equidistant from the sheet on opposite sides. With these facts established, Gauss's law gives us at once the field intensity, as follows: Draw a cylinder, as in Fig. 1.23, with P on one side and P' on the other, of cross-section area A. The outward flux is found only at the ends, so that if E_P denotes the magnitude of the field at P, and $E_{P'}$ the magnitude of P', the outward flux is $AE_P + AE_{P'} = 2AE_P$. The charge enclosed is σA. Hence $2AE_P = 4\pi\sigma A$, or

$$E_P = 2\pi\sigma \tag{28}$$

We see that the field strength is independent of r, the distance from the sheet. Equation 28 could have been derived more laboriously by calculating the vector sum of the contributions to the field at P from all the little elements of charge in the sheet.

The field of an infinitely long line charge, we found, varies inversely as the distance from the line, while the field of an infinite sheet has the same strength at all distances. These are simple consequences of the fact that the field of a point charge varies as the inverse square of the distance. If that doesn't yet seem compellingly obvious, look at it this way: Roughly speaking, the part of the line charge that is mainly responsible for the field at P, in Fig. 1.21, is the near part—the charge within a distance of order of magnitude r. If we lump all this together and forget the rest, we have a concentrated charge of magnitude $q \approx \lambda r$, which ought to produce a field proportional to q/r^2, or λ/r. In the case of the sheet, the amount of charge that is "effective," in this sense, increases proportionally to r^2 as we go out

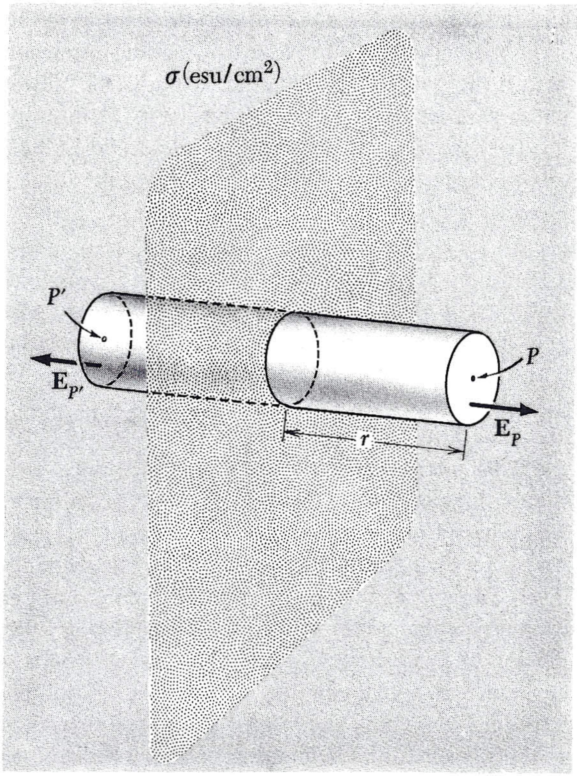


FIGURE 1.23
Using Gauss' law to find the field of an infinite flat sheet of charge.

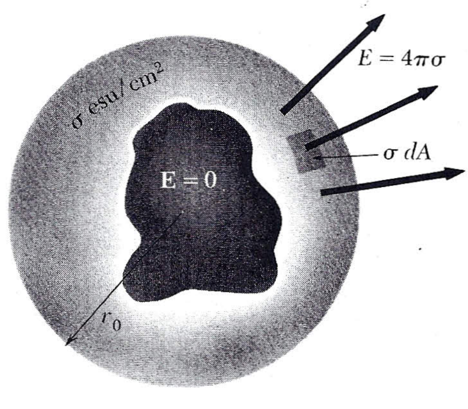
from the sheet, which just offsets the $1/r^2$ decrease in the field from any given element of charge.

THE FORCE ON A LAYER OF CHARGE

1.14 The sphere in Fig. 1.24 has a charge distributed over its surface with the uniform density σ, in esu/cm^2. Inside the sphere, as we have already learned, the electric field of such a charge distribution is zero. Outside the sphere the field is Q/r^2, where Q is the total charge on the sphere, equal to $4\pi r_0^2\sigma$. Just outside the surface of the sphere the field strength is $4\pi\sigma$. Compare this with Eq. 28 and Fig. 1.23. In both cases Gauss' law is obeyed: The *change* in E, from one side of the layer to the other, is equal to $4\pi\sigma$.

What is the electrical force experienced by the charges that make up this distribution? The question may seem puzzling at first because the field **E** arises from these very charges. What we must think about is the force on some small element of charge dq, such as a small patch of area dA with charge $dq = \sigma\, dA$. Consider, separately, the force on dq due to all the other charges in the distribution,

FIGURE 1.24
A spherical surface with uniform charge density σ.



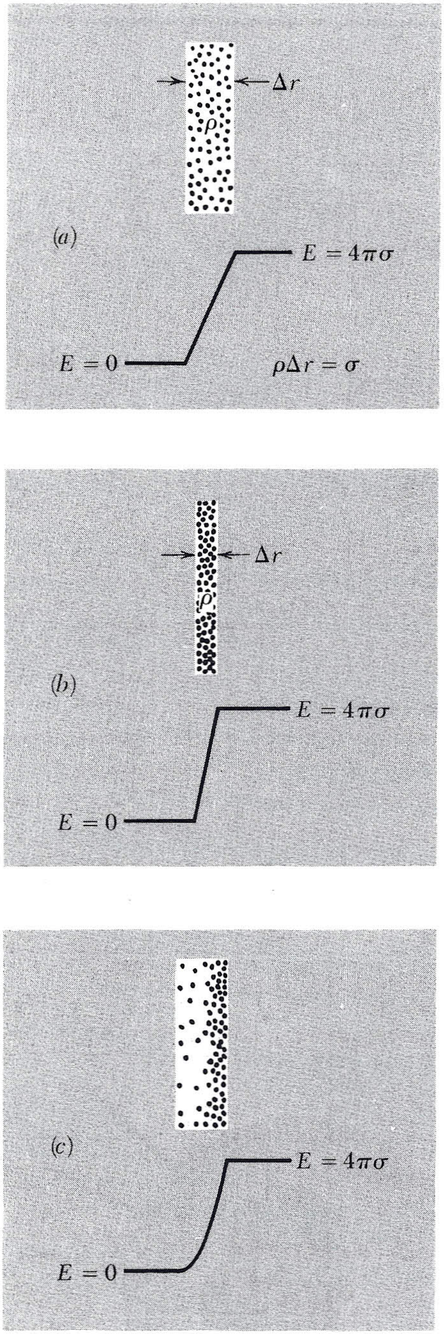


FIGURE 1.25
The net change in field at a charge layer depends only on the total charge per unit area.

and the force on the patch due to the charges within the patch itself. This latter force is surely zero. Coulomb repulsion between charges within the patch is just another example of Newton's third law; the patch as a whole cannot push on itself. That simplifies our problem, for it allows us to use the entire electric field **E**, *including* the field due to all charges in the patch, in calculating the force $d\mathbf{F}$ on the patch of charge dq:

$$d\mathbf{F} = \mathbf{E}\, dq = \mathbf{E}\, \sigma\, dA \tag{29}$$

But what E shall we use, the field $E = 4\pi\sigma$ outside the sphere or the field $E = 0$ inside? The correct answer, as we shall prove in a moment, is the *average* of the two fields.

$$dF = \tfrac{1}{2}(4\pi\sigma + 0)\, \sigma\, dA = 2\pi\sigma^2\, dA \tag{30}$$

To justify this we shall consider a more general case, and one that will introduce a more realistic picture of a layer of surface charge. Real charge layers do not have zero thickness. Figure 1.25 shows some ways in which charge might be distributed through the thickness of a layer. In each example the value of σ, the total charge per unit area of layer, is the same. These might be cross sections through a small portion of the spherical surface in Fig. 1.24 on a scale such that the curvature is not noticeable. To make it more general, however, we have let the field on the left be E_1 (rather than 0, as it was inside the sphere), with E_2 the field strength on the right. The condition imposed by Gauss's law, for given σ, is in each case

$$E_2 - E_1 = 4\pi\sigma \tag{31}$$

Now let us look carefully within the layer where the field is changing continuously from E_1 to E_2 and there is a volume charge density $\rho(x)$ extending from $x = 0$ to $x = x_0$, the thickness of the layer (Fig. 1.26). Consider a much thinner slab, of thickness $dx \ll x_0$, which contains per unit area an amount of charge $\rho\, dx$. The force on it is

$$dF = E\rho\, dx \tag{32}$$

Thus the total force per unit area of our charge layer is

$$F = \int_0^{x_0} E\rho\, dx \tag{33}$$

But Gauss's law tells us that dE, the change in E through the thin slab, is just $4\pi\rho\, dx$. Hence $\rho\, dx$ in Eq. 33 can be replaced by $dE/4\pi$, and the integral becomes

$$F = \frac{1}{4\pi}\int_{E_1}^{E_2} E\, dE = \frac{1}{8\pi}(E_2^2 - E_1^2) \tag{34}$$

Since $E_2 - E_1 = 4\pi\sigma$, the result in Eq. 34, after being factored, can be expressed as

$$F = \tfrac{1}{2}(E_1 + E_2)\sigma \tag{35}$$

We have shown, as promised, that for given σ the force per unit area on a charge layer is determined by the mean of the external field on one side and that on the other.† This is independent of the thickness of the layer, as long as it is small compared to the total area, and of the variation $\rho(x)$ in charge density within the layer.

The direction of the electrical force on an element of the charge on the sphere is, of course, outward whether the surface charge is positive or negative. If the charges do not fly off the sphere, that outward force must be balanced by some inward force, not included in our equations, which can hold the charge carriers in place. To call such a force "nonelectrical" would be misleading, for electrical attractions and repulsions are the dominant forces in the structure of atoms and in the cohesion of matter generally. The difference is that these forces are effective only at short distances, from atom to atom, or from electron to electron. Physics on that scale is a story of individual particles. Think of a charged rubber balloon, say, 10 cm in radius, with 20 esu of negative charge spread as uniformly as possible on its outer surface. It forms a surface charge of density $\sigma = 20/400\pi = 0.016$ esu/cm^2. The resulting outward force, per cm^2 of surface charge, is $2\pi\sigma^2$, or 0.0016 dynes/cm^2. In fact our charge consists of about 4×10^{10} electrons attached to the rubber film. As there are about 30 million extra electrons per cm^2, "graininess" in the charge distribution is hardly apparent. However, if we could look at one of these extra electrons, we would find it roughly 10^{-4} cm—an enormous distance on an atomic scale—from its nearest neighbor. This electron would be stuck, electrically stuck, to a local molecule of rubber. The rubber molecule would be attached to adjacent rubber molecules, and so on. If you pull on the electron, the force is transmitted in this way to the whole piece of rubber. Unless, of course, you pull hard enough to tear the electron loose from the molecule to which it is attached. That would take an electric field many thousands of times stronger than the field in our example.

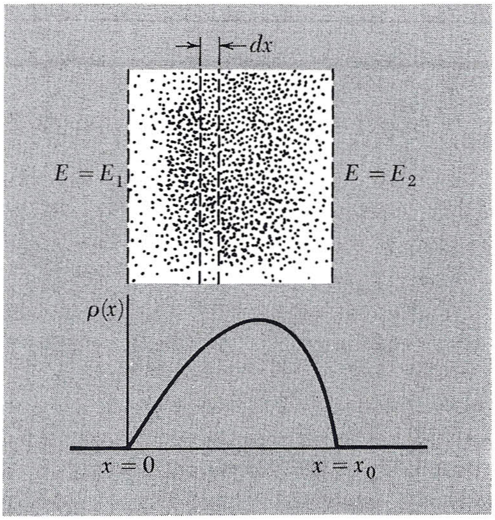
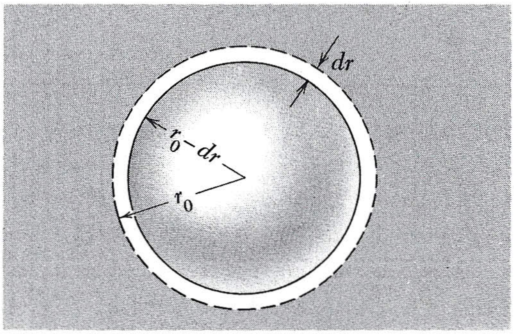


FIGURE 1.26
Within the charge layer of density $\rho(x)$, $E(x + dx) - E(x) = 4\pi\rho\, dx$.

FIGURE 1.27
Shrinking a spherical shell or charged balloon.



ENERGY ASSOCIATED WITH THE ELECTRIC FIELD

1.15 Suppose our spherical shell of charge is compressed slightly, from an initial radius of r_0 to a smaller radius, as in Fig. 1.27. This requires that work be done against the repulsive force, $2\pi\sigma^2$ dynes for

†Note that this is *not* necessarily the same as the average field within the layer, a quantity of no special interest or significance.

each square centimeter of surface. The displacement being dr, the total work done is $(4\pi r_0^2)(2\pi\sigma^2)\,dr$, or $8\pi^2 r_0^2 \sigma^2\,dr$. This represents an *increase* in the energy required to assemble the system of charges, the energy U we talked about in Section 1.5:

$$dU = 8\pi^2 r_0^2 \sigma^2\,dr \tag{36}$$

Notice how the electric field E has been changed. Within the shell of thickness dr the field was zero and is now $4\pi\sigma$. Beyond r_0 the field is unchanged. In effect we have created a field of strength $E = 4\pi\sigma$ filling a region of volume $4\pi r_0^2\,dr$. We have done so by investing an amount of energy given by Eq. 36 which, if we substitute $E/4\pi$ for σ, can be written like this:

$$dU = \frac{E^2}{8\pi} 4\pi r_0^2\,dr \tag{37}$$

This is an instance of a general theorem which we shall not prove now: *The potential energy U of a system of charges, which is the total work required to assemble the system, can be calculated from the electric field itself simply by assigning an amount of energy* $(E^2/8\pi)\,dv$ *to every volume element* dv *and integrating over all space where there is electric field.*

$$U = \frac{1}{8\pi}\int_{\substack{\text{Entire}\\ \text{field}}} E^2\,dv \tag{38}$$

E^2 is a scalar quantity, of course: $E^2 \equiv \mathbf{E}\cdot\mathbf{E}$.

One may think of this energy as "stored" in the field. The system being conservative, that amount of energy can of course be recovered by allowing the charges to go apart; so it is nice to think of the energy as "being somewhere" meanwhile. Our accounting comes out right if we think of it as stored in space with a density of $E^2/8\pi$, in ergs/cm^3. There is no harm in this, but in fact we have no way of identifying, quite independently of anything else, the energy stored in a particular cubic centimeter of space. Only the total energy is physically measurable, that is, the work required to bring the charge into some configuration, starting from some other configuration. Just as the concept of electric field serves in place of Coulomb's law to explain the behavior of electric charges, so when we use Eq. 38 rather than Eq. 9 to express the total potential energy of an electrostatic system, we are merely using a different kind of bookkeeping. Sometimes a change in viewpoint, even if it is at first only a change in bookkeeping, can stimulate new ideas and deeper understanding. The notion of the electric field as an independent entity will take form when we study the dynamical behavior of charged matter and electromagnetic radiation.

We run into trouble if we try to apply Eq. 38 to a system that contains a point charge, that is, a finite charge q of zero size. Locate q at the origin of the coordinates. Close to the origin E^2 will approach q^2/r^4. With $dv = 4\pi r^2\, dr$, the integrand $E^2\, dv$ will behave like dr/r^2, and our integral will blow up at the limit $r = 0$. That simply tells us that it would take infinite energy to pack finite charge into zero volume—which is true but not helpful. In the real world we deal with particles like electrons and protons. They are so small that for most purposes we can ignore their dimensions and think of them as point charges when we consider their electrical interaction with one another. How much energy it took to make such a particle is a question that goes beyond the range of classical electromagnetism. We have to regard the particles as supplied to us ready-made. The energy we are concerned with is the work done in moving them around.

The distinction is usually clear. Consider two charged particles, a proton and a negative pion, for instance. Let $\mathbf{E}_p$ be the electric field of the proton, $\mathbf{E}_\pi$ that of the pion. The total field is $\mathbf{E} = \mathbf{E}_p + \mathbf{E}_\pi$, and $\mathbf{E} \cdot \mathbf{E}$ is $E_p^2 + E_\pi^2 + 2\mathbf{E}_p \cdot \mathbf{E}_\pi$. According to Eq. 38 the total energy in the electric field of this two-particle system is

$$\begin{aligned} U &= \frac{1}{8\pi}\int E^2\, dv \\ &= \frac{1}{8\pi}\int E_p^2\, dv + \frac{1}{8\pi}\int E_\pi^2\, dv + \frac{1}{4\pi}\int \mathbf{E}_p \cdot \mathbf{E}_\pi\, dv \end{aligned} \tag{39}$$

The value of the first integral is a property of any isolated proton. It is a constant of nature which is not changed by moving the proton around. The same goes for the second integral, involving the pion's electric field alone. It is the third integral that directly concerns us, for it expresses the energy required to assemble the system *given* a proton and a pion as constituents.

The distinction could break down if the two particles interact so strongly that the electrical structure of one is distorted by the presence of the other. Knowing that both particles are in a sense composite (the proton consisting of three quarks, the pion of two), we might expect that to happen during a close approach. In fact, nothing much happens down to a distance of 10^{-13} cm. At shorter distances, for strongly interacting particles like the proton and the pion, nonelectrical forces dominate the scene anyway.

That explains why we do not need to include "self-energy" terms like the first two integrals in Eq. 39 in our energy accounts for a system of elementary charged particles. Indeed, we want to omit them. We are doing just that, in effect, when we replace the actual distribution of discrete elementary charges (the electrons on the rubber balloon) by a perfectly continuous charge distribution.

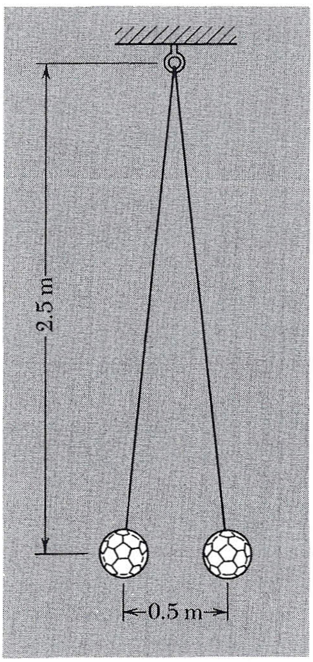
PROBLEMS

1.1 In the domain of elementary particles, a natural unit of mass is the mass of a *nucleon,* that is, a proton or a neutron, the basic massive building blocks of ordinary matter. Given the nucleon mass as 1.6×10^{-24} gm and the gravitational constant G as 6.7×10^{-8} cm^3/gm-sec^2, compare the gravitational attraction of two protons with their electrostatic repulsion. This shows why we call gravitation a very *weak* force. The distance between the two protons in the helium nucleus could be at one instant as much as 10^{-13} cm. How large is the force of electrical repulsion between two protons at that distance? Express it in newtons, and in pounds. Even stronger is the *nuclear* force that acts between any pair of hadrons (including neutrons and protons) when they are that close together.

1.2 On the utterly unrealistic assumption that there are no other charged particles in the vicinity, at what distance below a proton would the upward force on an electron (electron mass $\simeq 10^{-27}$ gm) equal the electron's weight?

1.3 Two volley balls, mass 0.3 kilogram (kg) each, tethered by nylon strings and charged with an electrostatic generator, hang as shown in the diagram. What is the charge on each in coulombs, assuming the charges are equal? (*Reminder:* the weight of a 1-kg mass on earth is 9.8 newtons, just as the weight of a 1-gm mass is 980 dynes.)

PROBLEM 1.3



1.4 At each corner of a square is a particle with charge q. Fixed at the center of the square is a point charge of opposite sign, of magnitude Q. What value must Q have to make the total force on each of the four particles zero? With Q set at that value, the system, in the absence of other forces, is in equilibrium. Do you think the equilibrium is stable?

Ans. $Q = 0.957q$.

1.5 A thin plastic rod bent into a semicircle of radius R has a charge of Q, in esu, distributed uniformly over its length. Find the strength of the electric field at the center of the semicircle.

1.6 Three positive charges, A, B, and C, of 3×10^{-6}, 2×10^{-6}, and 2×10^{-6} coulombs, respectively, are located at the corners of an equilateral triangle of side 0.2 meter.

(*a*) Find the magnitude in newtons of the force on each charge.

(*b*) Find the magnitude in newtons/coulomb of the electric field at the center of the triangle.

Ans. (*a*) 2.34 newtons on A, 1.96 newtons on B and C; (*b*) 6.74×10^5 newtons/coulomb.

1.7 Find a geometrical arrangement of one proton and two electrons such that the potential energy of the system is exactly zero. How many such arrangements are there with the three particles on the same straight line?

1.8 Calculate the potential energy, per ion, for an infinite one-dimensional ionic crystal, that is, a row of equally spaced charges of magnitude e and alternating sign. [*Hint:* The power-series expansion of ln $(1 + x)$ may be of use.]

1.9 A spherical volume of radius a is filled with charge of uniform density ρ. We want to know the potential energy U of this sphere of charge, that is, the work done in assembling it. Calculate it by building the sphere up layer by layer, making use of the fact that the field outside a spherical distribution of charge is the same as if all the charge were at the center. Express the result in terms of the total charge Q in the sphere.

Ans. $U = \frac{3}{5}(Q^2/a)$.

1.10 At the beginning of the century the idea that the rest mass of the electron might have a purely electrical origin was very attractive, especially when the equivalence of energy and mass was revealed by special relativity. Imagine the electron as a ball of charge, of constant volume density out to some maximum radius r_0. Using the result of Problem 1.9, set the potential energy of this system equal to mc^2 and see what you get for r_0. One defect of the model is rather obvious: Nothing is provided to hold the charge together!

1.11 A charge of 1 esu is at the origin. A charge of -2 esu is at $x = 1$ on the x axis.

(*a*) Find a point on the x axis where the electric field is zero.

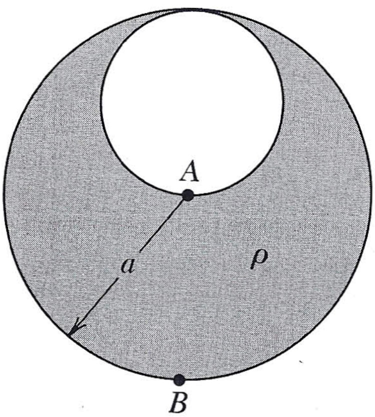
(*b*) Locate, at least approximately, a point on the y axis where the electric field is parallel to the x axis. [A calculator should help with (*b*).]

1.12 Another problem for your calculator: Two positive ions and one negative ion are fixed at the vertices of an equilateral triangle. Where can a fourth ion be placed so that the force on it will be zero? Is there more than one such place?

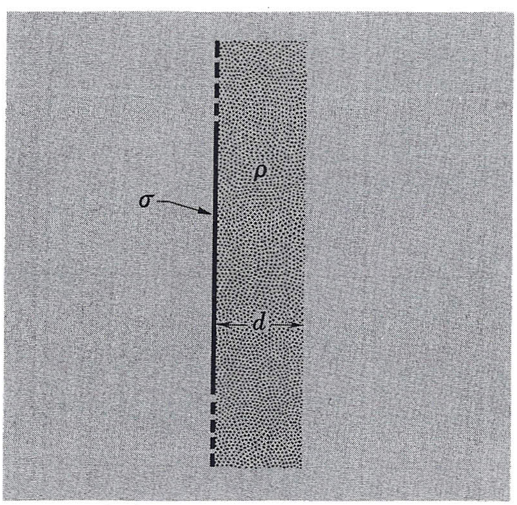
1.13 The passage of a thundercloud overhead caused the vertical electric field strength in the atmosphere, measured at the ground, to rise to 0.1 statvolt/cm.

(*a*) How much charge did the thundercloud contain, in esu per cm^2 of horizontal area?

(*b*) Suppose there was enough water in the thundercloud in the form of 1-millimeter (mm)-diameter drops to make 0.25 cm of rainfall, and that it was those drops which carried the charge. How large was the electric field strength at the surface of one of the drops?



PROBLEM 1.16



PROBLEM 1.19

1.14 A charge Q is distributed uniformly around a thin ring of radius b which lies in the xy plane with its center at the origin. Locate the point on the positive z axis where the electric field is strongest.

1.15 Consider a spherical charge distribution which has a constant density ρ from $r = 0$ out to $r = a$ and is zero beyond. Find the electric field for all values of r, both less than and greater than a. Is there a discontinuous change in the field as we pass the surface of the charge distribution at $r = a$? Is there a discontinuous change at $r = 0$?

1.16 The sphere of radius a was filled with positive charge at uniform density ρ. Then a smaller sphere of radius $a/2$ was carved out, as shown in the figure, and left empty. What are the direction and magnitude of the electric field at A? At B?

1.17 (a) A point charge q is located at the center of a cube of edge length d. What is the value of $\int \mathbf{E} \cdot d\mathbf{a}$ over one face of the cube?

(b) The charge q is moved to one corner of the cube. What is now the value of the flux of **E** through each of the faces of the cube?

1.18 Two infinite plane sheets of surface charge, of density $\sigma = 6$ esu/cm^2 and $\sigma = -4$ esu/cm^2, are located 2 cm apart, parallel to one another. Discuss the electric field of this system. Now suppose the two planes, instead of being parallel, intersect at right angles. Show what the field is like in each of the four regions into which space is thereby divided.

1.19 An infinite plane has a uniform surface charge distribution σ on its surface. Adjacent to it is an infinite parallel layer of charge of thickness d and uniform volume charge density ρ. All charges are fixed. Find E everywhere.

1.20 Consider a distribution of charge in the form of a circular cylinder, like a long charged pipe. Prove that the field inside the pipe is zero. Prove that the field outside is the same as if the charge were all on the axis. Is either statement true for a pipe of square cross section on which the charge is distributed with uniform surface density?

1.21 The neutral hydrogen atom in its normal state behaves in some respects like an electric charge distribution which consists of a point charge of magnitude e surrounded by a distribution of negative charge whose density is given by $-\rho(r) = Ce^{-2r/a_0}$. Here a_0 is the *Bohr radius,* 0.53×10^{-8} cm, and C is a constant with the value required to make the total amount of negative charge exactly e. What is the net electric charge inside a sphere of radius a_0? What is the electric field strength at this distance from the nucleus?

1.22 Consider three plane charged sheets, A, B, and C. The sheets are parallel with B below A and C below B. On each sheet there is surface charge of uniform density: -4 esu/cm^2 on A, 7 esu/cm^2 on B, and -3 esu/cm^3 on C. (The density given includes charge on both sides of the sheet.) What is the magnitude of the electrical force on each sheet, in dynes/cm^2? Check to see that the total force on the three sheets is zero.

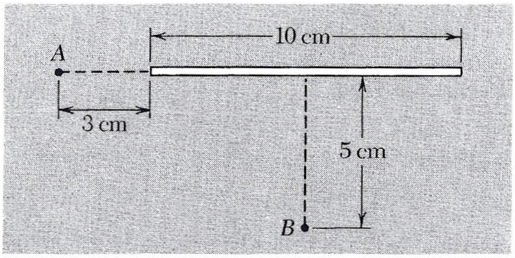
Ans. 32π dynes/cm^2 on A; 14π dynes/cm^2 on B; 18π dynes/cm^2 on C.

1.23 A sphere of radius R has a charge Q distributed uniformly over its surface. How large a sphere contains 90 percent of the energy stored in the electrostatic field of this charge distribution?

Ans. Radius: $10R$.

1.24 A thin rod 10 cm long carries a total charge of 8 esu uniformly distributed along its length. Find the strength of the electric field at each of the two points A and B located as shown in the diagram.

PROBLEM 1.24



1.25 The relation in Eq. 27 expressed in SI units becomes

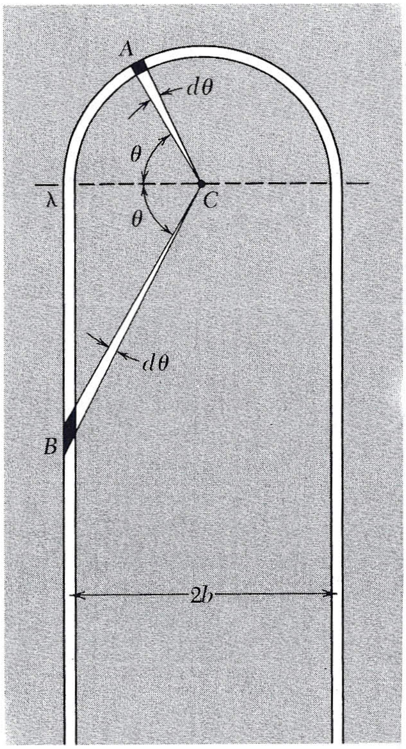
$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{r}$$

with r in meters, λ in coulombs/meter, and E in newtons/coulomb. Consider a high-voltage direct current power line which consists of two parallel conductors suspended 3 meters apart. The lines are oppositely charged. If the electric field strength halfway between them is 15,000 newtons/coulomb, how much excess positive charge resides on a 1-km length of the positive conductor?

Ans. 6.26×10^{-4} coulomb.

1.26 Two long, thin parallel rods, a distance $2b$ apart, are joined by a semicircular piece of radius b, as shown. Charge of uniform linear density λ is deposited along the whole filament. Show that the field $\mathbf{E}$ of this charge distribution vanishes at the point C. Do this by comparing the contribution of the element at A to that of the element at B which is defined by the same values of θ and $d\theta$.

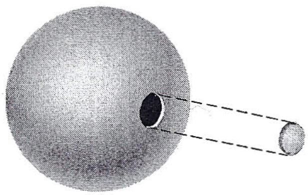
PROBLEM 1.26



1.27 An infinite chessboard with squares of side s has a charge e at the center of every white square and a charge $-e$ at the center of every black square. We are interested in the work W required to transport one charge from its position on the board to an infinite distance from the board, along a path perpendicular to the plane of the board. Given that W is finite (which is plausible but not so easy to prove), do you think it is positive or negative? To calculate an approximate value for W, consider removing the charge from the central square of a 7×7 board. (Only 9 different terms are involved in that sum.) Or write a program and compute the work to remove the central charge from a much larger array, for instance a 101×101 board. Comparing the result for the 101×101 board with that for a 99×99 board, and for a 103×103 board, should give some idea of the rate of convergence toward the value for the infinite array.

1.28 Three protons and three electrons are to be placed at the vertices of a regular octahedron of edge length a. We want to find the energy of the system, that is, the work required to assemble it starting with the particles very far apart. There are two essentially different arrangements. What is the energy of each?

Ans. $-3.879e^2/a$; $-2.121e^2/a$.



PROBLEM 1.29

1.29 The figure shows a spherical shell of charge, of radius a and surface density σ, from which a small circular piece of radius $b \ll a$ has been removed. What is the direction and magnitude of the field at the midpoint of the aperture? There are two ways to get the answer. You can integrate over the remaining charge distribution to sum the contributions of all elements to the field at the point in question. Or, remembering the superposition principle, you can think about the effect of replacing the piece removed, which itself is practically a little disk. Note the connection of this result with our discussion of the force on a surface charge—perhaps that is a third way in which you might arrive at the answer.

1.30 Concentric spherical shells of radius a and b, with $b > a$, carry charge Q and $-Q$, respectively, each charge uniformly distributed. Find the energy stored in the electric field of this system.

1.31 Like the charged rubber balloon described on page 31, a charged soap bubble experiences an outward electrical force on every bit of its surface. Given the total charge Q on a bubble of radius R, what is the magnitude of the resultant force tending to pull any hemispherical half of the bubble away from the other half? (Should this force divided by $2\pi R$ exceed the surface tension of the soap film interesting behavior might be expected!)

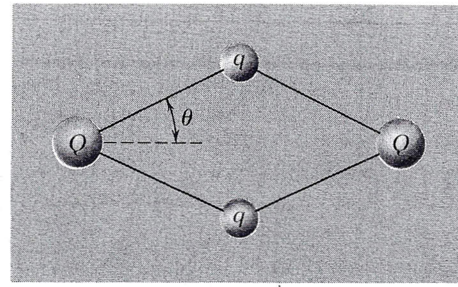
Ans. $Q^2/8R^2$.

1.32 Suppose three positively charged particles are constrained to move on a fixed circular track. If the charges were all equal, an equilibrium arrangement would obviously be a symmetrical one with the particles spaced 120° apart around the circle. Suppose that two of the charges are equal and the equilibrium arrangement is such that these two charges are 90° apart rather than 120°. What is the relative magnitude of the third charge?

Ans. 3.154.

1.33 Imagine a sphere of radius a filled with negative charge of uniform density, the total charge being equivalent to that of two electrons. Imbed in this jelly of negative charge two protons and assume that in spite of their presence the negative charge distribution remains uniform. Where must the protons be located so that the force on each of them is zero? (This is a surprisingly realistic caricature of a hydrogen molecule; the magic that keeps the electron cloud in the molecule from collapsing around the protons is explained by quantum mechanics!)

1.34 Four positively charged bodies, two with charge Q and two with charge q, are connected by four unstretchable strings of equal length. In the absence of external forces they assume the equilibrium configuration shown in the diagram. Show that $\tan^3 \theta = q^2/Q^2$. This can be done in two ways. You could show that this relation must hold if the total force on each body, the vector sum of string tension and electrical repulsion, is zero. Or you could write out the expression for the energy U of the assembly (like Eq. 7 but for four charges instead of three) and minimize it.



PROBLEM 1.34

1.35 Consider the electric field of two protons b cm apart. According to Eq. 1.38 (which we stated but did not prove) the potential energy of the system ought to be given by

$$U = \frac{1}{8\pi}\int \mathbf{E}^2\,dv = \frac{1}{8\pi}\int (\mathbf{E}_1 + \mathbf{E}_2)^2\,dv$$

$$= \frac{1}{8\pi}\int \mathbf{E}_1^2\,dv + \frac{1}{8\pi}\int \mathbf{E}_2^2\,dv + \frac{1}{4\pi}\int \mathbf{E}_1 \cdot \mathbf{E}_2\,dv$$

where $\mathbf{E}_1$ is the field of one particle alone and $\mathbf{E}_2$ that of the other. The first of the three integrals on the right might be called the "electrical self-energy" of one proton; an intrinsic property of the particle, it depends on the proton's size and structure. We have always disregarded it in reckoning the potential energy of a system of charges, on the assumption that it remains constant; the same goes for the second integral. The third integral involves the distance between the charges. The third integral is not hard to evaluate if you set it up in spherical polar coordinates with one of the protons at the origin and the other on the polar axis, and perform the integration over r first. Thus, by direct calculation, you can show that the third integral has the value e^2/b, which we already know to be the work required to bring the two protons in from an infinite distance to positions a distance b apart. So you will have proved the correctness of Eq. 38 for this case, and by invoking superposition you can argue that Eq. 38 must then give the energy required to assemble any system of charges.