Second-Order Phase Transitions: Modern Developments

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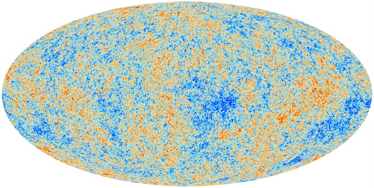
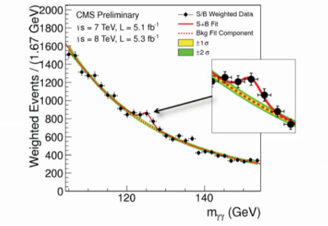
Quantum Field Theory is a universal mathematical structure that follows from two central pillars of modern physics

- Quantum Mechanics
- Special Relativity

It is the main framework in Elementary Particles, Statistical Mechanics, Condensed Matter, Stochastic Processes, and Cosmology.





There are many possible models in QFT. A fundamental question is to understand how they are related.

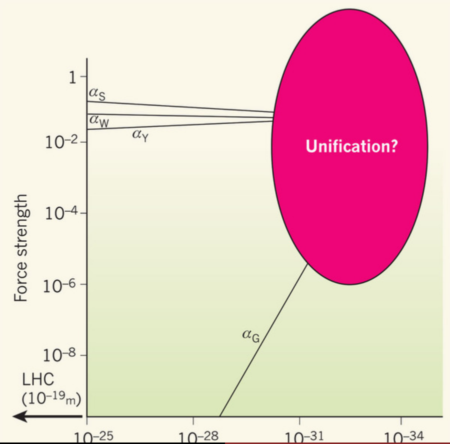
For example, we can start from some given model and deform the Feynman integral (or the Hamiltonian) by

$$\left\langle e^{\int d^d x \sum_i \lambda^i \mathcal{O}_i} \right\rangle$$

where the $\mathcal{O}_i$ are various functions of the degrees of freedom.

The most fundamental and confusing observation is that the λ^i are not really well defined numbers [Landau et al, Gell-Mann Low, Kadanoff, Wilson...]!!

Their actual numerical value depends on the resolution of the experiment.



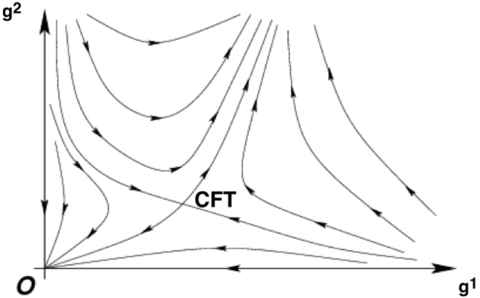
As we decrease the resolution (i.e. we coarse grain) the various options are

- Irrelevant: the coupling λ decreases as we go to long distances.
- Relevant: the coupling λ increases as we go to long distances.
- Exactly Marginal: the coupling does not change: it is a genuine number.

The last option is pretty rare but it appears in supersymmetric theories and in 2d models such as the Ashkin-Teller model.
Note: the irrelevant (relevant) couplings could decrease (increase) as a power law or logarithmically in the resolution.

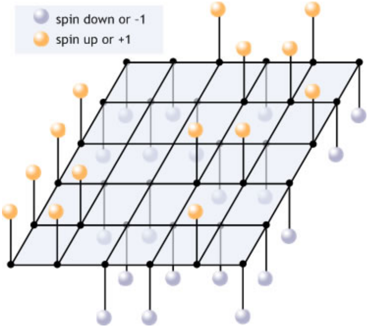
Typically: Only finitely many relevant couplings!

By removing all the relevant couplings by hand (turning knobs/fine tuning) we find a theory that at sufficiently long distances no longer depends on the resolution.



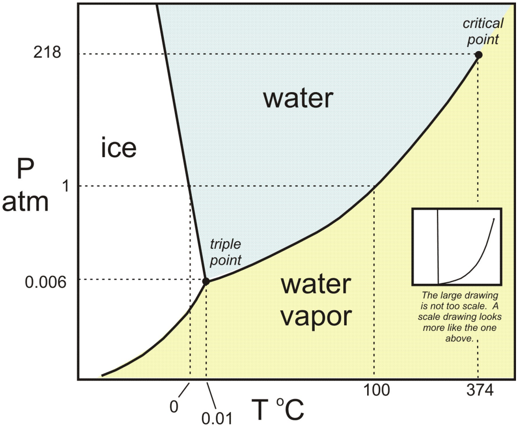
We call such points that are invariant under coarse graining *fixed points*. A more correct way to think about such flows and fixed points is as a gradient flow

Example 1: A Ferromagnet.

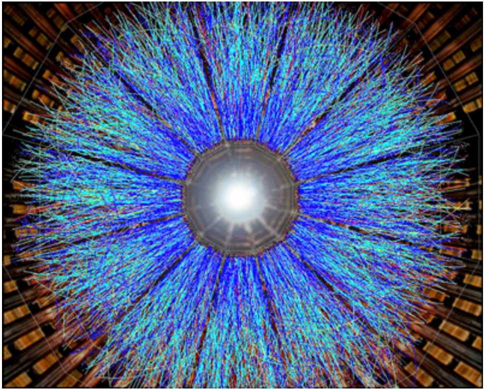


L^d spins with nearest-neighbor interaction energy $J > 0$ if they are misaligned.

Example 2: The water-vapor transition.



Example 3: The confinement-deconfinement transition in $SU(2)$ Yang-Mills theory.



Hadrons turn into a plasma of gluons.

All of these examples have at some T_c the **same** fixed point, which is the 3d Ising model! Long range correlations develop and micro structure becomes irrelevant.
Ginzburg-Landau theory:

$$H = \int d^d x \left(r(\nabla M)^2 + cM^2 + \lambda M^4 + ... \right)$$

The partition function

$$Z = \int [dM] e^{-H}$$

encodes all the thermodynamics properties at the phase transition.

Some experimentally interesting quantities are the exponents $\alpha, \beta, \gamma, \delta, \eta, \nu$

$$C \sim (T - T_c)^{-\alpha} \ , \qquad M \sim (T_c - T)^{\beta} \ , \qquad \chi \sim (T - T_c)^{-\gamma} \ ,$$

$$M \sim h^{1/\delta} \ , \qquad \langle M(\vec{n}) M(0) \rangle \sim \frac{1}{|\vec{n}|^{d-2+\eta}} \ , \qquad \xi \sim (T - T_c)^{-\nu} \ .$$

Amazingly, one discovers four relations between these 6 quantities:

$$\alpha + 2\beta + \gamma = 2 \ ,$$

$$\gamma = \beta(\delta - 1) \ ,$$

$$\gamma = \nu(2 - \eta) \ ,$$

$$\nu d = 2 - \alpha \ .$$

The explanation of this miracle is that at T_c the symmetry of the system is enhanced:

$$SO(3) \times \mathbb{R}^3 \to SO(3) \times \mathbb{R}^3 \times \mathbb{R}_+ \ ,$$

with

$$\mathbb{R}_+ : \ x \to \lambda x$$

and $\lambda \in \mathbb{R}_+$.

The dilation charge associated to $\mathbb{R}_+$ is called Δ. It can be diagonalized. If we have a local operator $\mathcal{O}$ in the theory, $\Delta(\mathcal{O})$ would uniquely determine its two-point correlator

$$\langle \mathcal{O}(n)\mathcal{O}(0)\rangle \sim \frac{1}{n^{2\Delta(\mathcal{O})}} \ .$$

Local operators could also have spin s, but we suppress it in the meantime.

We can make contact with our previous terminology:

- Relevant: $\Delta \leq 3$
- Irrelevant: $\Delta \geq 3$

(For $\Delta = 3$ a separate analysis of $\langle \mathcal{O}(n)\mathcal{O}(n')\mathcal{O}(n'')\rangle$ is necessary.)

The relevant operators appear in phase diagrams and the irrelevant ones disappear at long distances.

(irrelevant operators can be dangerously irrelevant and affect the phase diagram, but not the fixed point – e.g. perovskite materials)

In the 3d Ising model, there are two relevant operators: $M(x)$ and $\epsilon(x)$.

$$\Delta_M = 0.518... \ , \qquad \Delta_\epsilon = 1.413...$$

- Ferromagnet: magnetic field and temperature.
- water-vapor: pressure and temperature.
- SU(2) Yang-Mills: fundamental quark mass and temperature.

These two numbers are "fundamental constants."

The four miraculous relations among $\alpha, \beta, \gamma, \delta, \eta, \nu$ can be simply understood from scale invariance:

$$\alpha = \frac{d - 2\Delta_\epsilon}{d - \Delta_\epsilon} ,$$

$$\beta = \frac{\Delta_M}{d - \Delta_\epsilon} ,$$

$$\gamma = \frac{d - 2\Delta_M}{d - \Delta_\epsilon} ,$$

$$\delta = \frac{d - \Delta_M}{\Delta_M} ,$$

$$\eta = 2 - d + 2\Delta_M ,$$

$$\nu = \frac{1}{d - \Delta_\epsilon} .$$

This is how these relations were originally explained.

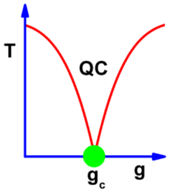
There has been a lot of recent progress based on the observation that the symmetry is actually **bigger!!**

$$SO(3) \times \mathbb{R}^3 \to SO(3) \times \mathbb{R}^3 \times \mathbb{R}_+ \to SO(4,1)$$

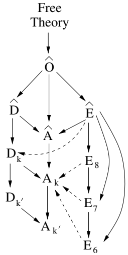
Theories with this big symmetry are called (3d) **Conformal Field Theories** (CFTs). There are many such theories. The 3d Ising model is perhaps the simplest nontrivial example. [Polyakov?!]

Applications of Conformal Field Theories:

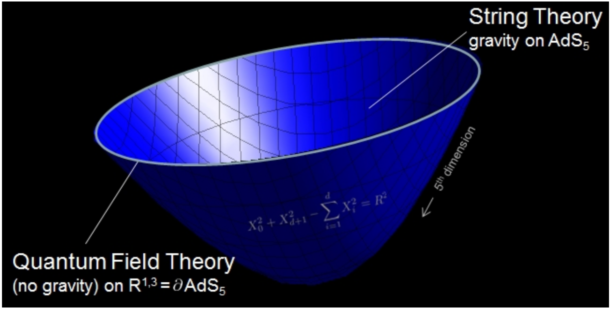
- Quantum phase transitions



- Fixed points of the Renormalization Group Flow.



Quantum gravity in AdS_{d+1} is described by a boundary Conformal Field Theory.



$SO(d+1,1)$ is the conformal group that acts on $\mathbb{R}^d$. This is the set of all transformations that preserve orthogonal lines.
It consists of

$$d(d-1)/2 \text{ rotations}$$

$$d \text{ translations}$$

$$1 \text{ dilation } (\Delta)$$

$$d \text{ special conformal transformations}$$

The last d symmetry generators are **beyond** Ginzburg-Landau theory.

Essentially all the examples that we know of second order phase transitions which have

$$SO(d) \times \mathbb{R}^d \times \mathbb{R}_+$$

have the full $SO(d+1,1)$. It has been even verified experimentally and in Monte Carlo simulations in some examples.

$d = 2$: An argument from the 80's by Joe Polchinski. The argument is a spin-off of Zamolodchikov's renormalization-group irreversibility theorem.

$d = 4$: With the advent of the irreversibility theorem in 2011-2012, it was possible to give an argument in $d = 4$ for this mysterious symmetry enhancement.

$d = 3$: Open question. Seems to hold in all known examples. Perhaps can be approached using tools of entanglement entropy?

What are the applications and implication of this surprising symmetry enhancement? we are still trying to understand. But we already know quite a bit. My goal here is to describe briefly some of the things we have learned.

The d bonus special conformal transformations act on space as

$$x^i \to \frac{x^i - b^i x^2}{1 - 2b \cdot x + b^2 x^2} ,$$

where b^i is any vector in $\mathbb{R}^d$.
It turns out that these are enough to fix three-point correlation functions (for primary operators) as follows

$$\langle \mathcal{O}_1(n')\mathcal{O}_2(n)\mathcal{O}_3(0)\rangle = \frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{n^{\Delta_2+\Delta_3-\Delta_1}(n-n')^{\Delta_1+\Delta_2-\Delta_3} n'^{\Delta_1+\Delta_3-\Delta_2}}$$

Remember

$$\langle \mathcal{O}_i(n)\mathcal{O}_j(0)\rangle \sim \frac{\delta_{ij}}{n^{2\Delta_i(\mathcal{O})}} .$$

SO(d+1,1) symmetry therefore fixes the two- and three-point functions in terms of a collection of numbers

$$\Delta_i \qquad , \qquad C_{ijk}$$

which are called the "CFT data."

It turns out that ALL the correlation functions are fixed in terms of the CFT data.

How can we say anything useful about this collection of numbers ?

$$\{\Delta_i\} \quad , \quad \{C_{ijk}\}$$

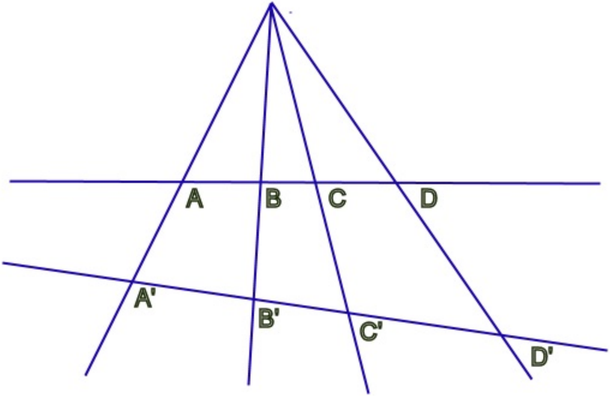
This collection is generally infinite. It is easiest to measure the relevant, low dimension, operators (such as M, ϵ), but also the others exist and are in principle measurable.

Consider a four-point function with operators at n_1, n_2, n_3, n_4. We can form conformally invariant ratios:

$$u = \frac{n_{12}^2 n_{34}^2}{n_{14}^2 n_{23}^2} \ , \qquad v = \frac{n_{13}^2 n_{24}^2}{n_{13}^2 n_{24}^2}$$

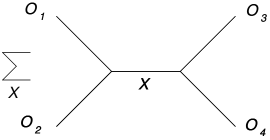
and the general four-point function is

$$\langle \mathcal{O}_1(n_1)\mathcal{O}_2(n_2)\mathcal{O}_3(n_3)\mathcal{O}_4(n_4)\rangle \sim F(u,v) \ .$$



$$\frac{(CA / CB)}{(DA / DB)} = \frac{(C'A' / C'B')}{(D'A' / D'B')}$$

We represent the four point function as a sum over infinitely many three point functions:



$$\sim \sum_X C_{12X} C_{X34} G(\Delta_{1,2,3,4,X}, u, v)$$

Therefore,

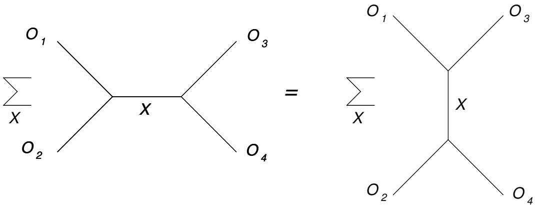
$$F(u,v) \sim \sum_X C_{12X} C_{X34} G(\Delta_{1,2,3,4,X}, u, v)$$

The functions G are partial waves (generalizations of the Legendre polynomials). Thus, once we know the CFT data, the four-point function can be in principle computed.

The dynamics is in saying that we can make the decomposition in two different ways. And we get (roughly speaking)

$$\sum_X C_{12X} C_{X34} G(\Delta_{1,2,3,4,X}, u, v) = \sum_X C_{13X} C_{X24} G(\Delta_{1,3,2,4,X}, v, u)$$

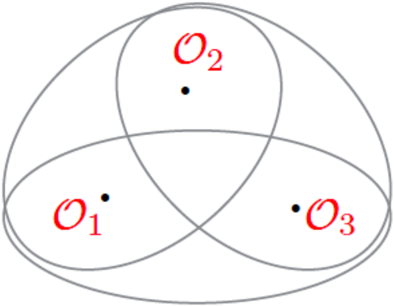
This equation is supposed to determine/constrain the allowed Δ_i and C_{ijk} that can furnish legal conformal theories.



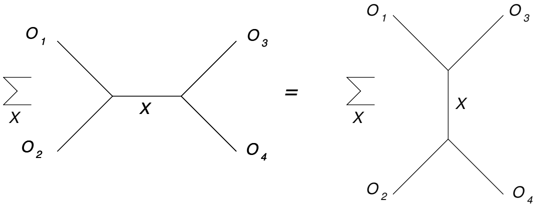
This is extremely surprising:

Maybe we could classify all the possible conformal theories by just solving self-consistency algebraic equations.

For the mathematically oriented: these equations are similar to the equations of associative rings. It is also very similar in spirit to the classification of Lie Algebras. One may also be reminded of the classification of Topological Field Theories.

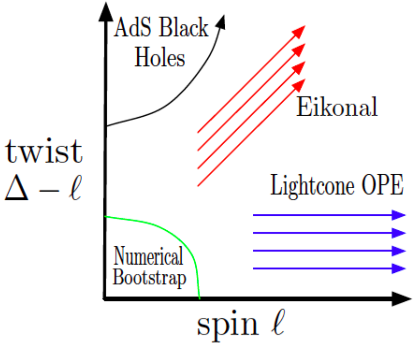


Therefore, if we had a better idea about what the equations



imply, that would be useful in many branches of physics. More ambitiously, we could hope to classify all the solutions!

Recently, there has been dramatic progress on this problem both from the analytic and numeric points of view.



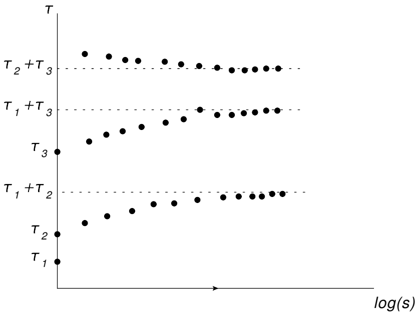
I will quote, without proofs, four very general analytic results. They are numerically and even experimentally testable.

Result I: Additivity of the Spectrum

If we have (Δ_1, s_1) and (Δ_2, s_2) in the spectrum, then there are operators (Δ_i, s_i) which have $\Delta_i - s_i$ arbitrarily close to $\Delta_1 - s_1 + \Delta_2 - s_2$.

Typically, to find operators with $\Delta_i - s_i \sim \Delta_1 - s_1 + \Delta_2 - s_2$ we would need to take large Δ_i, s_i.

This leads to a rather peculiar spectrum, schematically as follows



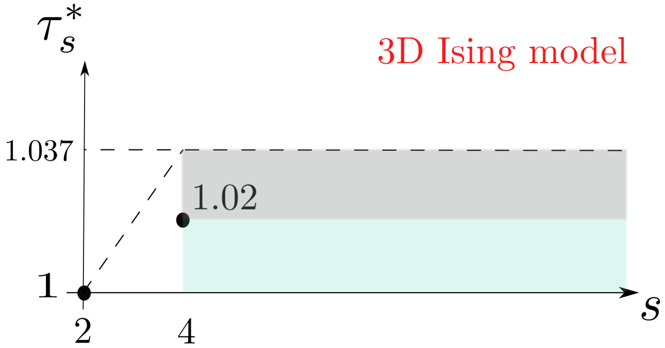
(In the figure, $\tau \equiv \Delta - s$.)

Note that the spectrum of many famous 2d systems such as the Ising and Potts models is not as complicated. This is because in the proof of the additivity theorem it is assumed that $d > 2$.

Result II: Convexity

$\Delta_i - s_i$ approaches the limiting value $\Delta_1 - s_1 + \Delta_2 - s_2$ in a convex manner.

These already lead to nontrivial results for the 3d Ising model. Since $\Delta_M = 0.518...$, there needs to be a family of operators with $\Delta - s$ approaching 1.037... from below, in a convex fashion:



This is beautifully verified by the measurement of the spin-4 operator dimension. Recently verified for spin 6. The rest awaits confirmation.

The approach to the asymptote is happening at large spin, i.e. small angular resolution. It turns out that in an appropriate sense

Result III: Weak Coupling at Small Angles

The approach to the asymptote is under analytic, perturbative control, even in strongly coupled CFTs!

For example, the leading deviation from the asymptote at large spin is

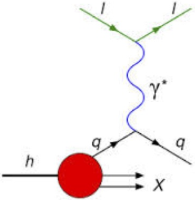
$$2\tau - \frac{d^2\Gamma(d+2)\Delta^2\Gamma^2(\Delta)}{2c_T(d-1)^2\Gamma^2(d/2+1)\Gamma^2(\Delta-d/2+1)}\frac{1}{s^2}+\cdots$$

This is like a one-loop calculation around the weakly coupled "$s=\infty$" point.

This leading-order computation has been much elaborated upon, improved and specialized to various situations. Using the inversion formula of Caron-Huot, it has been possible to make this large spin expansion even more systematic and include non-perturbative corrections at large spin.

So far we have mostly discussed the spectrum Δ_i. What about the C_{ijk} (i.e. the three-point functions)? The measurements of the structure constants C_{ijk} are really scarce. But using the numerical methods of the bootstrap (pioneered by Rattazzi, Rychkov, Tonni and Vichi) there are now quite precise predictions, for instance, in the 3d Ising model, $C_{\epsilon\epsilon\epsilon} = 1.532435(19)$.

On the theoretical front, it turns out that considering deep-inelastic scattering gedanken experiments we find interesting constraints



$$\langle T_{\mu\nu} O_{\mu_1 \ldots \mu_s} O_{\nu_1 \ldots \nu_s} \rangle > 0$$

for many of the tensor structures that appear in these three-point functions. This leads to many inequalities that can be explicitly tested. One can also understand these inequalities in the framework of conformal collider physics (Hofman-Maldance) and more recently similar ideas were applied at finite temperature.

Conclusions

- Conformal Field Theories are abundant in physics.
- They are determined by an intricate self-consistency condition. We still don't know much about the general consequences of this self-consistency condition.
- There are some results though on monotonicity, convexity, and additivity of the spectrum of dimensions. Also results about three-point functions.
- Many verifications in numerics and experiment.

Some Big Questions

- Conformal windows of Quantum Chromodynamics and QED_3? [applications for deconfined criticality, duality, and Technicolor theories]
- Large N CFTs may sometimes have a good approximation in terms of a weakly coupled theory in a higher-dimensional AdS space. Why does this happen and under what circumstances?
- The renormalization group connects different CFTs:

$$CFT_{UV} \to CFT_{IR}$$

  This leads to a foliation of the space of CFTs via the c, f, a monotonic functions. What about $d > 4$?
- Can there be nontrivial critical models with $d > 6$? how about $d = \infty$?

Thank you for the attention!