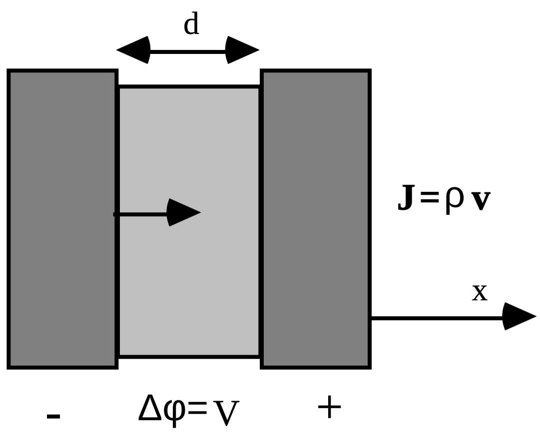
Exercise 1. In high current, high voltage devices, in order to find the condition of *space charge-limited flow,* it is useful to approximate the accelerating region as a pair of electrodes separated by a flat vacuum gap, as is shown below. When the current flow is activated, electrons are emitted by the cathode (-) and are accelerated freely in the electric field until they reach the anode, which they strike, and thus the free current is terminated.



The arrow shows the direction of motion of the accelerated electrons. Assume the system can be treated as one-dimensional, that is the gap distance d is much smaller than the plate width. Assume that the flow is static, with no parameters changing in time. All of the electrons will have the same velocity v (in the x-direction) at given point in x.

Since the flow is static, we have that $\rho(x) = J / v(x)$, with J constant.

a) Assume that the electrons are emitted with no kinetic energy at the cathode (at $x = 0$), which we choose to be at zero potential ($\phi(0) = 0$). Show that the (nonrelativistic) velocity can be written as $v(x) = \sqrt{2e\phi(x)/m_e}$.

b) Show that the (one-dimensional) Poisson equation in this case can be written in mks units as

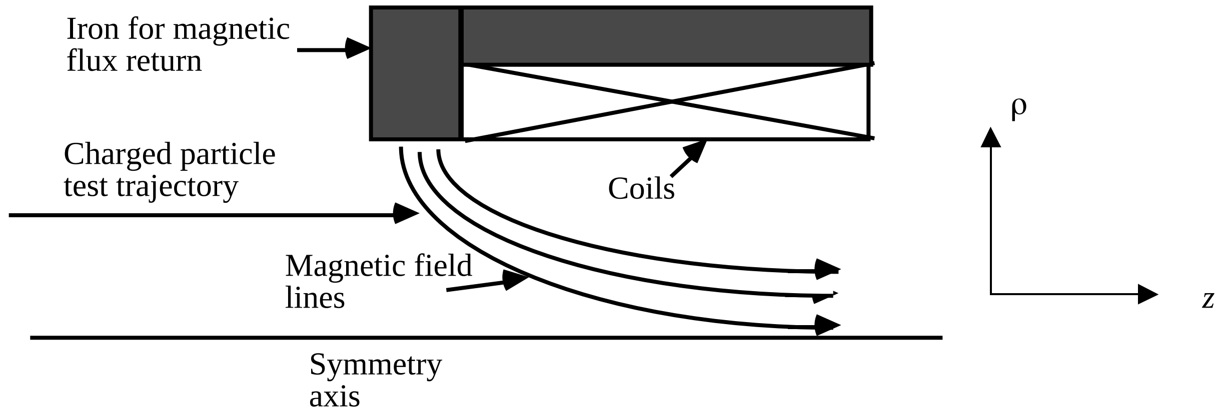
$$\frac{\partial^2\phi}{\partial x^2} = \sqrt{\frac{m_e}{2e}}\frac{|J|}{\varepsilon_0}\phi^{-1/2}.$$

c) The maximum allowable current density $|J|$ can be derived by solving this differential equation, subject to these boundary conditions: $\frac{\partial\phi}{\partial x} = 0$ at $x = 0$, that is, the electric field at the cathode is zero due to the charge buildup, stopping the acceleration of the flow, and of course $\phi(d) = V$. Show that the maximum current in this device (known as the Child-Langmuir limit) is of the form

$$I \quad V^{3/2},$$

and find the constant of proportionality (the perveance) between the current and $V^{3/2}$.

d) **Extra credit**: How far can you go towards solving this system with the correct relativistic relation between energy and velocity?



Exercise 2. In order to help you derive Busch's theorem, we refer to the figure above. The radial fringe field crossed with the longitudinal particle motion gives rise to an angular kick. This field can be approximated through use of Ampere's law

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho = -\frac{\partial B_z}{\partial z}$$

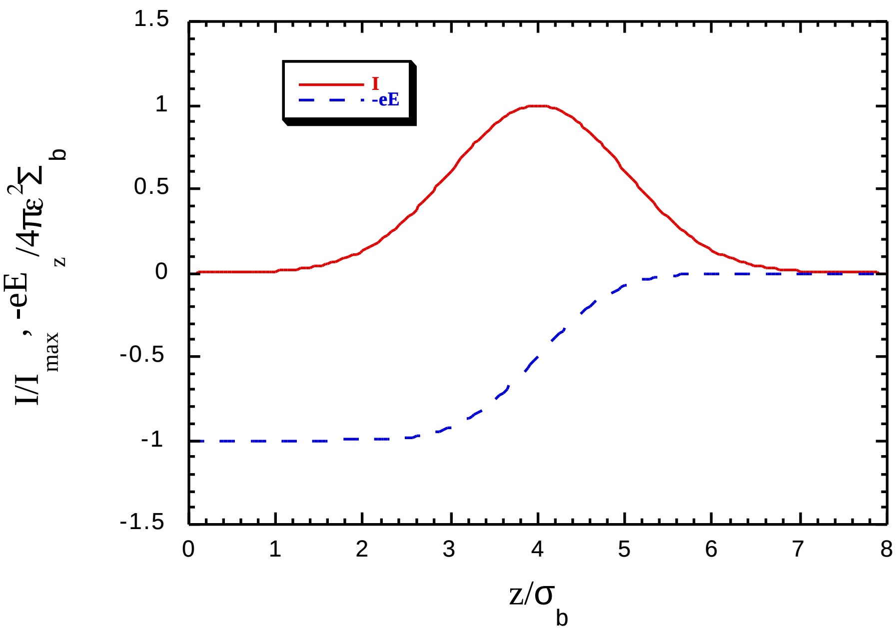
a) Assuming B_z is independent of ρ near the axis, find $B_\rho(\rho, z)$ as a function of $B_z(z)$.

b) Now integrate the angular kick received during a region where the field is changing,

$$\Delta p_\phi \quad q\int_{t_1}^{t_2} v_z B_\rho dt = q\int_{z_1}^{z_2} B_\rho dz\,.$$

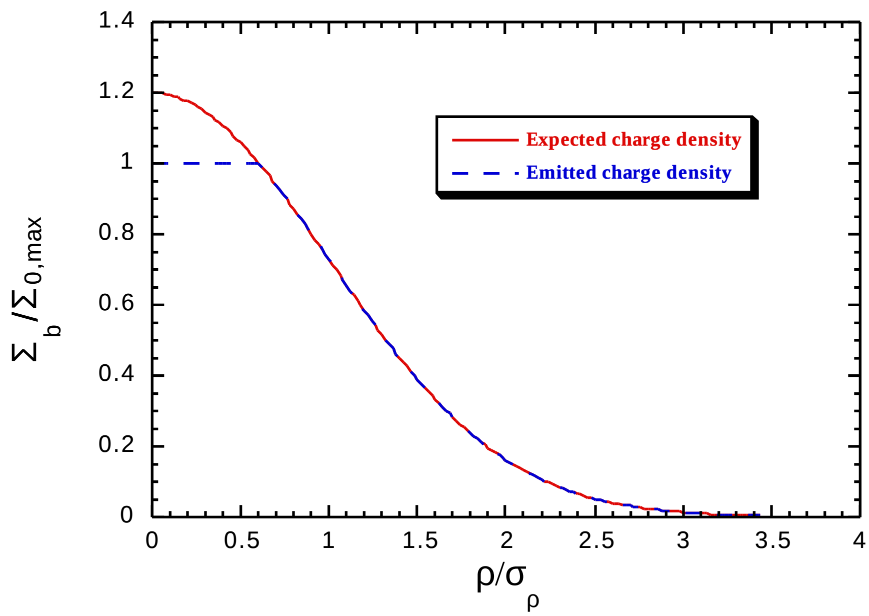
You should obtain a very general relation dependent only on the change in B_z. Show that Eq. 4.4 is valid under the assumptions given.

Exercise 3. This problem shows a transient version of the Child-Langmuir law governing maximum current, a law an maximum extracted charge, which is relevant to short pulses in photoinjectors. Because of the existence of image charges at the cathode, and because the beam is very short as it is emitted ($\sigma_z = v_b \sigma_t$), a retarding longitudinal force builds up during the short pulse, rising to be $F_z = -eE_z = 4\pi e^2 \Sigma_b$ (cgs units) at the back of the beam. Here Σ_b is the number of electrons per unit area emitted, and the field as a function of *z* s illustrated below. We have for this example assumed a Gaussian dependence of the emitted beam current on time; the field is simply proportional to the integral of this current (charge) distribution as one moves towards the cathode.



(a) For the LCLS injector parameters, $N_{e-} = 6.2 \times 10^9$ (1 nC), a=1 mm, calculate the (uniform) surface charge density of the emitted beam, and the retarding force at the back of the bunch. What is the maximum charge one can emit before completely canceling the applied field at injection? For calculations, it helps to use $e^2 = r_e m_e c^2$, where the classical radius of the electron is $r_e = 2.82 \times 10^{-15}$.

(b) Now look at the case where we use a laser beam with a Gaussian radial intensity distribution, $\mathbf{I}(\rho)=\mathbf{I}_0\exp\left(-\rho^2/2\sigma_\rho^2\right)$, so the expected surface charge density is also a radial Gaussian, $\Sigma_b(\rho) = \Sigma_0\exp\left(-\rho^2/2\sigma_\rho^2\right)$, where Σ_0 is obviously the maximum expected surface charge density. The surface charge density can not, however, exceed $\Sigma_{0,\max} = eE_0/4\pi e^2$, because the extracting electric field is cancelled, leading to an emission profile which looks like the picture below.



Assuming that $\Sigma_0 > \Sigma_{0,\max}$, so there is a flat part of the emitted charge density at $\Sigma_{0,\max}$, find an analytical expression for the total charge emitted, as a function of the expected charge in the absence of this limiting effect.