Physics 575 – Accelerator Physics and Technologies for Linear Colliders

(Winter 2002)

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Chapter 2 Basic Beam Dynamics

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2.1 Beam description

The goal of this chapter is to provide basic beam dynamics background useful for more detailed linear collider discussions.

2.1.1 Coordinates

We consider high-energy electron beams:

$$E = mc^2 \gamma \qquad (\gamma \approx 5 \times 10^5 \text{ for } E = 250 \text{ GeV})$$
$$p = mc\beta\gamma, \qquad \beta = \frac{v}{c} \approx 1 - \frac{1}{2\gamma^2} \approx 1. \tag{2.1}$$

Beams in high-energy accelerators are usually in the forms of a sequence of "bunches" moving along the accelerator axis. A bunch consists of electrons specified by the phase space coordinates $(x, x', y, y', z, \delta)$.

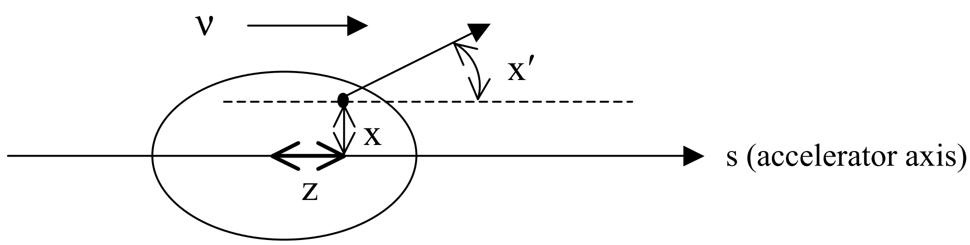


Figure 2.1

$$\begin{aligned}
s &= \text{position of the bunch center along the accelerator axis} \\
x,y &= \text{transverse coordinates} \\
x' &= \frac{dx}{ds}, \quad y' = \frac{dy}{ds} \qquad\qquad (2.2) \\
z &= \text{position of a particle relative to the beam center} \\
\delta &= \frac{p - p_o}{p_o} \approx \frac{E - E_o}{E_o}
\end{aligned}$$

These variables are equivalent to the canonical phase space variables (x, p_x, y, p_y, z, p_z).

The angles are very small, of the order of a few tens of microradians (10^{-5}). Thus the paraxial approximation can be used (x'^2 and y'^2 may be neglected).

2.1.2 Beam moments and emittances

Beam distribution in phase space is often of Gaussian shape and is completely described by second order beam moments:

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle}: && \text{rms beam size in x-direction} \\
\sigma_y &= \sqrt{\langle y^2 \rangle}: && \text{rms beam size in y-direction} \\
\sigma_{x'} &= \sqrt{\langle x'^2 \rangle}: && \text{rms beam angular divergence in x} \\
\sigma_{y'} &= \sqrt{\langle y'^2 \rangle}: && \text{rms beam angular divergence in y} \\
\sigma_z &= \sqrt{\langle z^2 \rangle}: && \text{rms bunch length} \\
\sigma_\delta &= \sqrt{\langle \delta^2 \rangle}: && \text{rms momentum spread}
\end{aligned} \tag{2.3}$$

There are also correlation moments

$$\langle xx' \rangle, \quad \langle xy \rangle, \quad \langle z\delta \rangle, \quad \text{etc.} \tag{2.4}$$

Beams are pictorially represented by phase space ellipses:

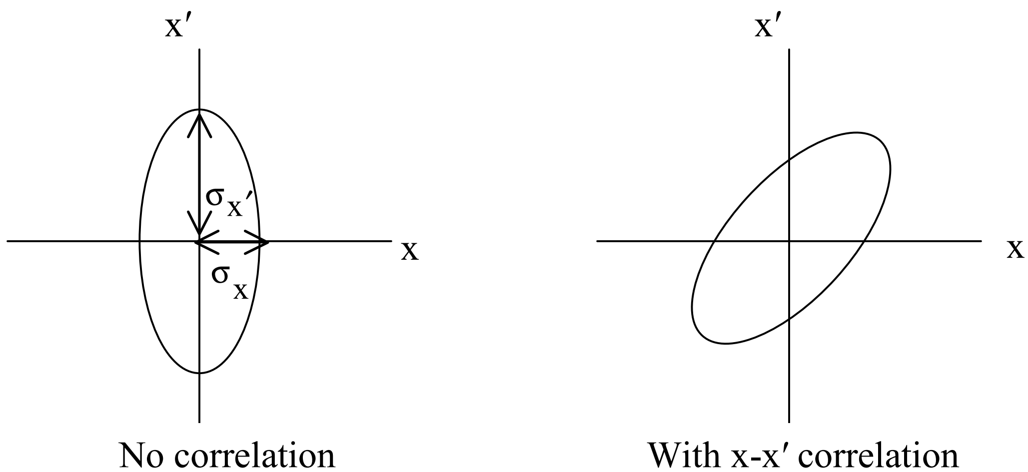


Figure 2.2

The phase space area is referred to as the "emittance." In the absence of correlations, the rms emittance is given by

$$\begin{aligned}
\varepsilon_x &= \sigma_x \sigma_{x'} \\
\varepsilon_y &= \sigma_y \sigma_{y'} \\
\varepsilon_z &= \sigma_z \sigma_\delta
\end{aligned} \tag{2.5}$$

In the past there have been confusions (sometimes very serious!) about the numerical factors π, 4, … etc., but the above convention is becoming the standard especially for electron accelerators.

2.1.3 Luminosity

Consider a bunch of electron beams colliding with a bunch of positrons moving the opposite direction:

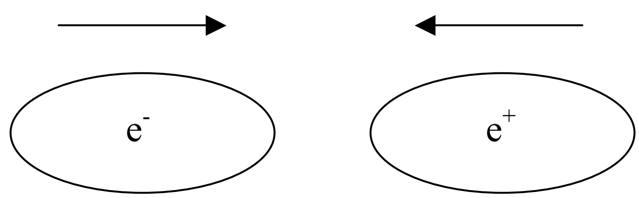


Figure 2.3

Let $\sigma_{e^+e^- \to X}$ be the cross section that an e^+e^- collision produces a particular final state X such as Higg's particles. The event rate for X is roughly as follows:

$$\text{Event rate} = N_- \bullet \sigma_{e^+e^- \to X} \bullet \frac{N_+}{A} \bullet f \tag{2.6}$$

$N_\pm$:	# of positrons (electrons) in each bunch
A:	transverse area of the beam
f:	repetition rate

The quantity multiplying the collision cross section is called the *luminosity* **L**.

$$\mathbf{L} \approx \frac{N_+ N_-}{A} f \tag{2.7}$$

More accurate calculation with Gaussian beams yield

$$\mathbf{L} = \frac{N_+ N_-}{4\pi\sigma_x\sigma_y} f. \tag{2.8}$$

Exercise: Derive the luminosity formula and discuss the luminosities of NLC and TESLA.

For NLC (one of the many versions), $N_+ = N_- = 0.75 \times 10^{10}$, $\sigma_x = 300$ nm, $\sigma_y = 6$ nm, $f = 180 \times 90 = 1.6 \times 10^4\ s^{-1}$. The corresponding luminosity is $\mathbf{L} \approx 0.4 \times 10^{34}$ $cm^{-2}s^{-1}$. The *geometric* luminosity is often enhanced by electromagnetic interaction of the colliding bunches by an enhancement factor $\mathbf{H}_D$ which is about 1.5 in this case, making the effective luminosity $\mathbf{L} = 0.55 \times 10^{34}\ cm^{-2}s^{-1}$. The integrated luminosity per year becomes, assuming there are 10^7 seconds in a year ("the Snow Mass year"), $5.5 \times 10^{41} cm^{-2}$, which is also 55 $(f\ barn)^{-1}$. Thus there will be about 55 events for a process with a cross section of 1 f barn.

2.1.4 Bunch evolution in free space – the need for focusing

A bunch may be in a tight, Gaussian shape at a certain location such as the collision point. What happens to the bunch if we let it evolve freely without any focusing device? **Answer:** The beam will spread out quickly and will be lost at wrong places.

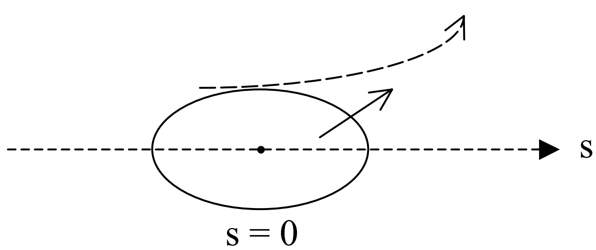


Figure 2.4

Let's consider a bunch at s = 0. An ith electron in the bunch has the transverse coordinate and angle $x_i(0)$ and $x_i'(0)$. Moving to a distance s, the coordinate becomes

$$x_i(s) = x_i(0) + s x_i'(0)$$
$$x_i'(s) = x_i'(0)$$

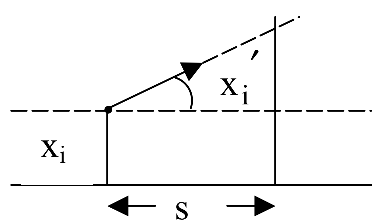


Figure 2.5

Thus the beam moments becomes (<…> = average)

$$\left\langle x^2 \right\rangle_s = \left\langle x_i^{\,2}(s) \right\rangle = \left\langle x_i^{\,2}(0) + 2x_i(0)x_{i'}(0) + s^2 x_i^{\,2'}(0) \right\rangle \tag{2.9}$$
$$= \left\langle x^2 \right\rangle_0 + s^2 \left\langle x'^2 \right\rangle_0 . \quad \text{(Assuming no correlation at s = 0)}$$

Thus the beam size increases due to the angular spread

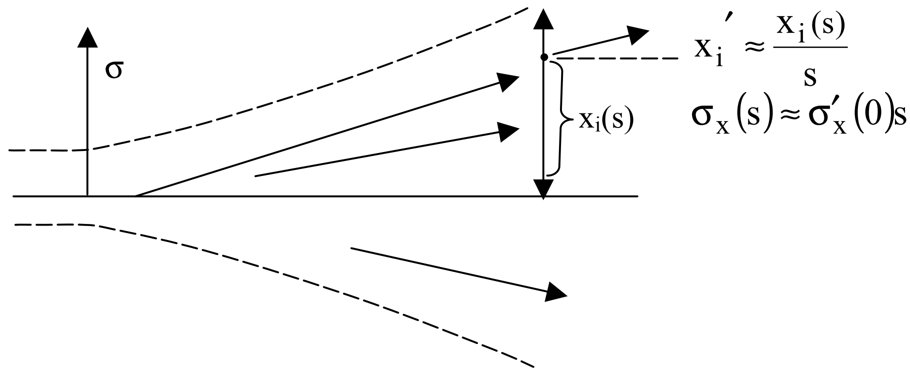


Figure 2.6

At large s $\left(\sigma_x'(0)s >> \sigma_x(0)\right)$, the angle and coordinate becomes correlated as is clear from the above figure, or

$$\left\langle x_i(s)x_i'(s) \right\rangle = s\left\langle x_i^{\,2}(0) \right\rangle. \tag{2.10}$$

Using the phase space diagram

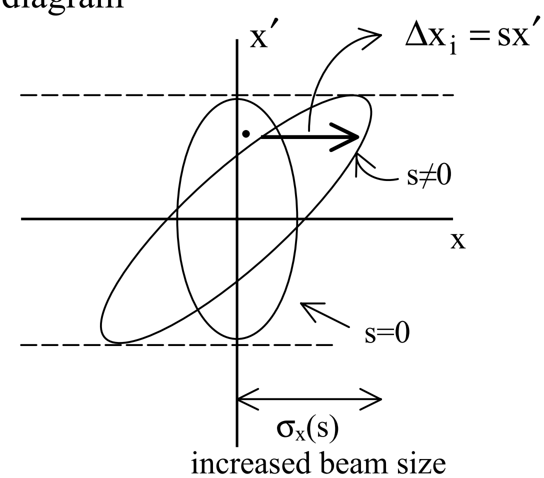


Figure 2.7

The phase space area is no longer the product of σ_x and $\sigma_{x'}$ due to the correlation. In the presence of correlation, the rms emittance is defined to be

$$\varepsilon_x(s) = \sqrt{\langle x^2 \rangle_s \langle x'^2 \rangle_s - \langle xx' \rangle_s^{\,2}} \tag{2.11}$$

It is easy to show that the emittance so defined is invariant. Thus the emittance is a measure of the inherent quality of the beam, independent of the beam propagation properties.

2.1.5 Beam-envelope function or the "Beta" function

The s-dependence of the rms beam size can be parameterized by introducing a function $\beta_x(s)$. Please try not to confuse β_x with particle speed $\beta = v/c$!

$$\sigma_x(s) = \sqrt{\varepsilon_x \beta_x(s)} = \sqrt{\sigma_x^{\,2}(0) + \sigma_{x'}^{\,2}(0) s^2} \tag{2.12}$$

Since $\varepsilon_x = \sigma_x(0)\sigma_x'(0)$, we have

$$\begin{aligned} \beta_x(s) &= \frac{\sigma_x(0)}{\sigma_{x'}(0)} + \frac{\sigma_{x'}(0)}{\sigma_x(0)} s^2 \\ &= \beta_x^{\,*} + \frac{s^2}{\beta_x^{\,*}} \end{aligned} \tag{2.13}$$

Here we have written the β-function at s = 0 as $\beta_x{*} = \beta_x(0)$, which is a standard notation for the beta function at the collision point. $\beta_y(s)$ and $\beta_y{*}$ are also introduced in a similar way. For $s = F >> \beta_x{*}$, Eq. (2.13) becomes

$$\beta_x(F) \approx \frac{F^2}{\beta_x{*}} \tag{2.14}$$

For the NLC $\beta_y{*} \approx 0.2\ \text{mm}$, the beam size at the first quadrupole about 1 m away is

$$\sigma_y(1\mathrm{m}) = \sigma_y(0)\sqrt{1+\left(\frac{1\mathrm{m}}{\beta_y *}\right)^2} \approx \sigma_y(0)\cdot 500. \tag{2.15}$$

That is the beam size increases by a factor of 500 to $\sigma_y = 3 \times 10^{-6}$ m. Although this is still small, beams cannot in general be allowed to expand forever, but must be focused back to remain in channel. The beam envelope function or the "beta function" is the property of the external focusing arrangement. This is in contrast to the emittance that characterizes the beam properties.

In a linear collider, one normally requires

$$\beta_y^* \gtrsim \sigma_z;$$

where σ_z is the rms bunch length. If this condition is violated, the beam density changes significantly during collisions leading to degradation in the luminosity. This is called the hourglass effect.

2.2 *Transverse Motion*

We have seen that beams must be focused. Beams must also be deflected sometimes. Beam trajectories in high-energy accelerators are controlled by magnets rather than electrodes since the magnetic forces are stronger.

2.2.1 Dipoles for deflection

In a dipole field B, the particle trajectory is a circle of radius $\rho = p/eB$ (Larmor radius). A sector dipole can therefore deflect particles:

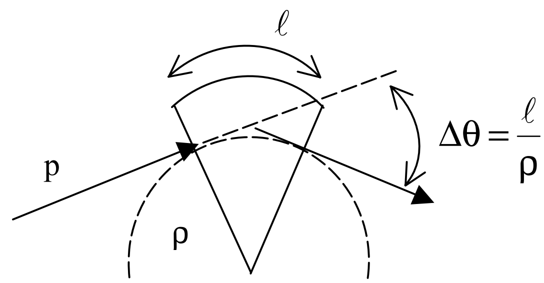
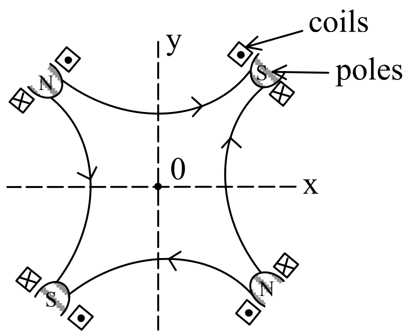


Figure 2.8

Bending is necessary in, for example, circular accelerators where $\Delta\theta$ must add up to 2π!

2.2.2 Equation of motion in quadrupoles

In a quadrupole, four poles of alternating polarities are placed symmetrically about the beam center:



(beam path is perpendicular to the page)

Figure 2.9

The field vanishes at origin; $B_x = B_y = 0$ at $x = y = 0$. Looking at the field lines in the figure, we see that the first order term in the expansion near the origin will be

$$B_y = \left(\frac{\partial By}{\partial x}\right)_o x, \quad B_x = \left(\frac{\partial Bx}{\partial y}\right)_o y \tag{2.16}$$

From Maxwell's equation, $(\nabla\times\mathbf{B}) = 0$, we derive that the coefficients in the above expansions are the same:

$$G = \left(\frac{\partial By}{\partial x}\right)_o = \left(\frac{\partial Bx}{\partial y}\right)_o \tag{2.17}$$

The equation for the transverse momentum components $(p_x, p_y) = \mathbf{p}_\perp$ is

$$\frac{d}{dt}\mathbf{p}_\perp = e(\mathbf{v}\times\mathbf{B})_\perp \tag{2.18}$$

Now,

- $\mathbf{v}$ in the RHS may be approximated by c $\mathbf{e}_z$ ($\mathbf{e}_z$ = unit vector along z)
- $\frac{d}{dt} \approx c\frac{d}{ds}$
- β and γ do not change in the static magnetic field
- $x' = \frac{dx}{ds}, \quad y' = \frac{dy}{ds}.$

The equation of motion in quadrupoles becomes then:

$$\frac{d^2}{ds^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -Kx \\ Ky \end{pmatrix} \tag{2.19}$$

where $K = \frac{eG}{p}$ $(p = m\gamma v \approx mc\gamma)$

2.2.3 Constant focusing

For K > 0 and constant, the x-motion is sinusoidal:

$$\frac{d^2}{ds^2}x \equiv x'' = -Kx$$

$$x = A\cos\left(\sqrt{K}s + \phi\right) \tag{2.20}$$

$$x' = -\sqrt{K}A\sin\left(\sqrt{K}s + \phi\right)$$

For a random distribution of A and ϕ, the beam is a collection of sinusoidal trajectories:

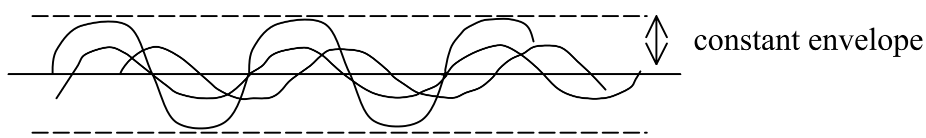


Figure 2.10

The beam envelope is constant:

$$\sigma_x{}^2 = \frac{1}{2}\left\langle A^2 \right\rangle = \text{const}$$

$$\sigma_{x'}{}^2 = K\sigma_x{}^2 \tag{2.21}$$

With the definition of the emittance ε_x and beam envelope function β_x, we have

$$\varepsilon_x = \sigma_x\sigma_{x'} = \frac{\left\langle A^2 \right\rangle}{2}\sqrt{K}$$

$$\beta_x = \sigma_x{}^2/\varepsilon_x = 1/\sqrt{K} \tag{2.22}$$

$$\therefore \left[\begin{array}{l} \left\langle A^2 \right\rangle = 2\varepsilon_x\beta_x \\ K = 1/\beta_x{}^2 \end{array} \right.$$

This is a well-focused beam in the x-direction. However, the only problem is that it is defocusing in the y-direction!

2.2.4 Discrete quadrupoles and the strong focusing principle

Fortunately, we will find that a periodic arrangement (lattice) of quadrupoles of alternating signs can focus a beam in both x and y directions over a long distance.

To simplify the discussion we make the thin lens approximation, namely that particles are deflected without changing the displacement.

$$x'' = \frac{d^2}{ds^2} x = \frac{d}{ds} x' = -Kx \tag{2.23}$$

Integrate once assuming K and x are constants:

$$\begin{aligned} \Delta x' &= x'_f - x'_i = -\frac{x}{F} \\ \Delta x &= x_f - x_i = 0 \end{aligned} \tag{2.24}$$

Here F = KΔs is the focal length.

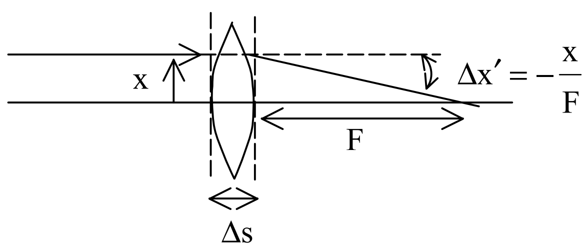
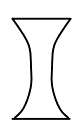


Figure 2.11

The quadrupole is focusing in the x-direction if $F > 0$. However, the same quadrupole in the y-direction will be defocusing. By convention, we mean a focusing quadrupole to be focusing in the x-direction. A defocusing quadrupole is represented by



We can show that the following periodic arrangement of focusing quadrupoles will keep the beam focused in the x-direction.

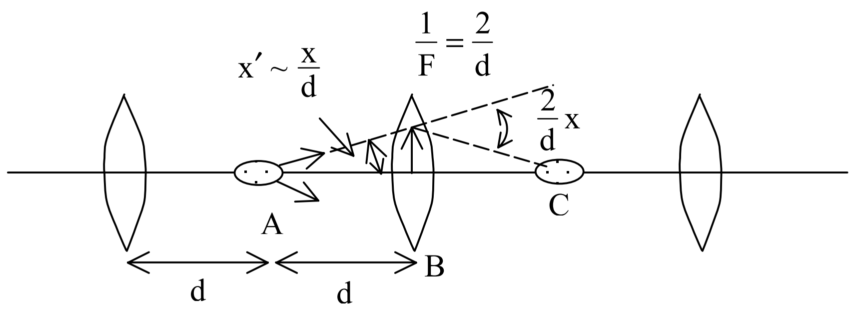


Figure 2.12

Distance between quadrupole = 2d. The focal length F = d/2. Halfway between the quads (A) the beam is focused to a spot similar as in the discussion of section 2.1.2. Assuming d >> $\sigma_x/\sigma_{x'}$, the beam at the quadrupole B develops x-x′ correlation, i.e., $x' \approx x/d$. Since the angular deflection at the lens is $\Delta x' = 2x/d$ for a lens of focal length F = d/2, the beam will focus back at the mirror point C. The pattern will then repeat itself.

The envelope function would look as follows

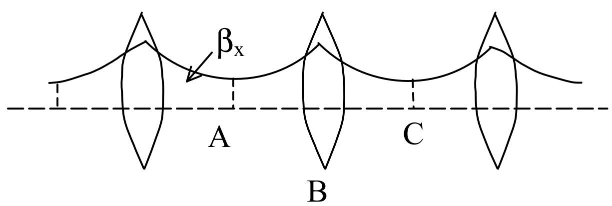


Figure 2.13

The beam phase space diagram

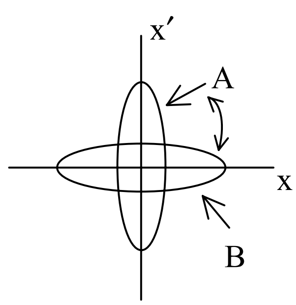


Figure 2.14

The beam phase space oscillates between a tall ellipse A (small size and large angular divergence) to a flat ellipse B (the phase space area being conserved).

However, we still have a problem in the y-direction, since the focusing has an opposite sign and will therefore be defocusing.

What will happen if we place a quadrupole of equal strength but opposite sign at waist locations?

Quadrupoles at waists do not have much effect on beam since the quadrupole action is proportional to the size which is small at waists. Thus the beam profile in x is the same as before.

However, we see now that the focusing properties in the y-direction are identical to the x-direction except it is displaced by one step d. The beam envelopes in the x- and y-directions will look as follows:

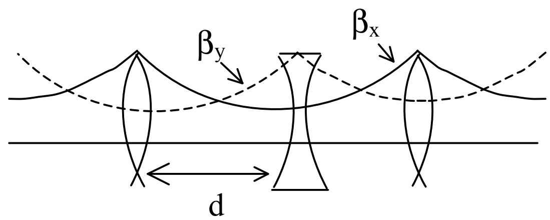


Figure 2.15

The beam envelopes in the x- and y-directions are out of step. Nevertheless, the beam is focused in both directions! The beam can now be transported to a long distance.

A periodic array of focusing quad (F), free space (O), defocusing quad (D), …, is called a FODO lattice, which is a very versatile beam transport device.

The fact that a combination of focusing and defocusing lenses can provide a net focusing was known to light optics. Application of this *strong focusing principle* to the quadrupole optics to obtain beam focusing in both directions was a critical contribution to the development of particle accelerators (Christopfilos, 1950, Courant and Snyder, 1952).

2.2.5 Betatron oscillation and phase advance

The trajectory of an individual electron in the FODO lattice discussed in Section 2.2.4 looks as follows:

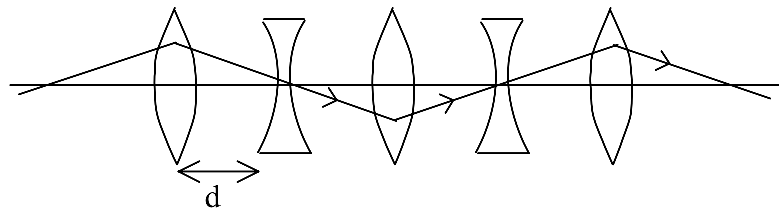


Figure 2.16

The motion is pseudo-sinusoidal with a period 4d. Note that the magnetic lattice is periodic with period 2d (between two Fs). The pseudo-sinusoidal motion is referred to as *the betatron* motion. An important quantity in the betatron motion is the phase advance per lattice period μ. In the case shown above $\mu = \pi$.

For a more rigorous analysis of the betatron motion one starts from the equation of motion:

$$x'' + K(s)x = 0 \tag{2.25}$$

where K(s) is periodic; K(s) = K(s+L). A general solution to such an equation can be written in the form (Floquet's theorem)

$$\begin{aligned} x &= \sqrt{2\varepsilon_x \beta_x(s)} \cos(\phi(s) + \phi_0,) \\ \phi(s) &= \int_0^s ds \frac{1}{\beta_x(s)} \end{aligned} \tag{2.26}$$

The envelope function β_x is the *periodic* solution of the following nonlinear equation:

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 K = 1. \tag{2.27}$$

The phase advance per period is given by

$$\mu = \int_0^L ds \frac{1}{\beta(s)}. \tag{2.28}$$

For the special case of K = const, these equations reduce to those in Section 2.2.3.

The phase advance normalized by 2π, $\nu=\mu/2\pi$, is known as the *tune*.

If we introduce the so-called Floquet variables (θ, u) via

$$u = \frac{1}{\sqrt{\beta(s)}} x \tag{2.29}$$

$$\theta = \frac{1}{\nu} \phi(s) \tag{2.30}$$

Then Eq. (2.25) reduces to a simple harmonic motion:

$$\frac{d^2 u}{d\theta^2} + \nu^2 u = 0 \tag{2.31}$$

For K = 0, the solution of Eq. (2.27) is of the form

$$\beta(s) = \beta_o + \frac{(s - s_o)^2}{\beta_o}. \tag{2.32}$$

This solution was discussed in section (2.1.4) with the special choice $s_o=0$. For a thin lens located at $s=s_F$, Eq. (2.27) can be integrated once to obtain the discontinuous charge of the derivative β':

$$(\Delta\beta')_F = -2\beta_F F \tag{2.33}$$

where $\beta_F = \beta(s_F)$. Equations (2.32) and (2.33) and the periodic boundary condition together completely specify the envelope function for a periodic lattice consisting of thin lenses and free spaces.

We should now make a small refinement to the discussion of Section 2.2.4; we have neglected the action of the quadrupoles in waist locations. This is true only if the beam size vanishes at waist, which is impossible for a non-vanishing emittance.

With correct beam dynamics calculations for finite beam sizes, it can be shown that a stable beam propagation in FODO lattice is possible if $d/2F < 1$. The betatron phase advance in this case is given by

$$\sin\frac{\mu}{2} = \frac{d}{2F}. \tag{2.34}$$

The maximum and minimum beta functions are

$$\beta_{\substack{max \\ min}} = 2d\left(\frac{1 \pm \sin\mu/2}{\sin\mu}\right). \tag{2.35}$$

2.2.6 Beam matching

Beams must enter a transport channel with correct initial conditions so that the stable periodic pattern derived in previous paragraphs is reproduced. For example, a beam entering to an FODO channel beginning with an F-quadrupole must have the ratio in the x- and y-spotsizes, $\sigma_x / \sigma_y = \sqrt{\beta_{max}/\beta_{min}}$ (assuming that $\varepsilon_x = \varepsilon_y$). This requirement is referred to as the "matching." If a beam is not matched, it will in general produce additional oscillations and eventually beam filamentation, a blur-up in phase space distribution and an effective emittance increase due to nonlinearities.

Errors in the quadrupole strengths in a linac will cause beams to become mismatched. Thus quadrupole errors must be tightly controlled to avoid an increase in the effective emittance.

Designing a suitable profile for beta functions often requires an experienced beam optics expert and is the basic step in particle accelerator design. Note that manipulation with the envelope function is an example of non-imaging optics.

2.2.7 Equation of motion including dipole field and energy error

The motion in a dipole is circular. The transverse displacement of a displaced circle measured from the reference circle will be sinusoidal with a period of $2\pi\rho$:

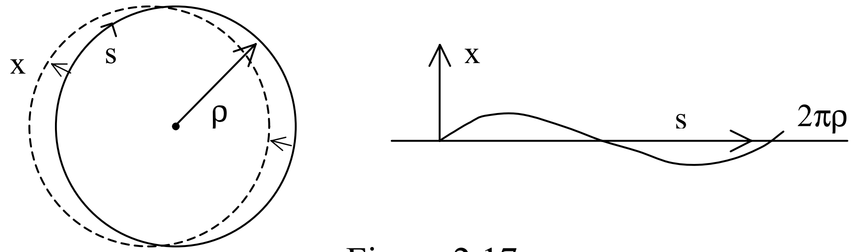


Figure 2.17

The equation of motion in dipole for small displacement is thus similar to a focusing quadrupole.

$$x'' + \frac{x}{\rho^2} = 0 \tag{2.36}$$

The equation of motion in the presence of both dipoles and quadrupoles is

$$\begin{aligned} x'' + \left(\frac{1}{\rho^2} + K\right)x = 0 \\ y'' - K = 0 \end{aligned} \tag{2.37}$$

For strong focusing machines, K>> $1/\rho^2$.

Consider now a particle with a slightly larger momentum p than the reference momentum p_o.

$$\delta = \frac{p - p_o}{p_o} \neq 0 \tag{2.38}$$

First the quadrupole strength will be reduced to (1-δ)K. This change produces a correction that is second order. More importantly, a momentum error produces an orbit displacement in a bending magnet.

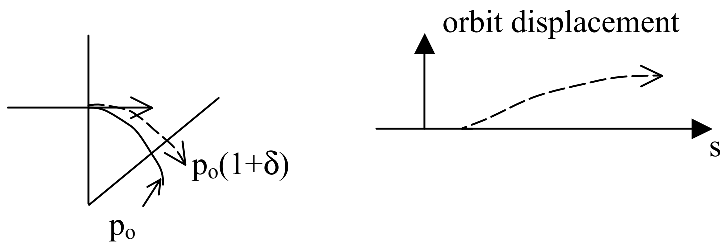


Figure 2.18

This displacement proportional to δ and must be added to the sinusoidal motion. Thus the x-displacement can be decomposed into two parts:

$$x = x_\beta + \eta_x \delta \tag{2.39}$$

The function η_x is called the *dispersion*. In general dispersions should not be too large since they contribute to the beam sizes proportional to the beam momentum spread.

Quadrupoles displaced transversely produce dipole fields and generate dispersion. Thus the quadrupole displacement in a linac must be tightly controlled to minimize the residual dispersion and beam size increases due to the momentum spread.

Our convention is that the x-direction is in the horizontal plane where bendings occur. In the vertical direction the dispersion is usually negligible. The equation for dispersion is

$$\eta_x'' + \left(\frac{1}{\rho^2} + K\right)\eta_x = \frac{1}{\rho}. \tag{2.40}$$

For constant focusing case discussed in Section 2.2.3, the solution is ($\beta_x \ll \rho$)

$$\eta_x = \frac{1}{\rho\left(K + 1/\rho^2\right)} \approx \frac{\beta_x^{\ 2}}{\rho}. \tag{2.41}$$

2.2.8 Chromatic effect of the final focus

Accelerated beams in linear colliders are brought to the interaction point (IP) by means of a strong lens (a quadrupole combination) to a tight spot for high luminosity collisions. The relations between the beam size σ and the angular spread σ' before the lens to the beam size σ_* and the angular spread σ_*' at the IP can be understood from the following figure:

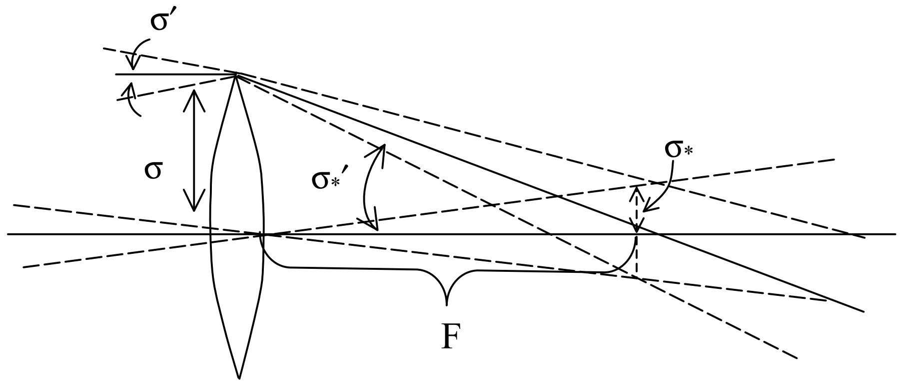


Figure 2.19

The beam before the lens is almost parallel. Thus the distance from the lens to the IP is the focal length F. From the figure it is apparent that

$$\sigma = F\sigma_*' \,, \sigma_* = F\sigma' \tag{2.42}$$

From this it follows that $\sigma\sigma' = \sigma_*\sigma'_*$, that is, the emittance is conserved. Also

$$\beta \equiv \frac{\sigma}{\sigma'} = F^2 \frac{\sigma_*'}{\sigma_*} \equiv \frac{F^2}{\beta_*} \tag{2.43}$$

Note that we have rederived Eq. (2.14)

The focal length of a quadrupole is proportional to the particle momentum p. Thus the IP spotsize will be larger for a beam with a non-vanishing momentum spread $\delta = \Delta p/p$. The increase can be estimated by the following figure:

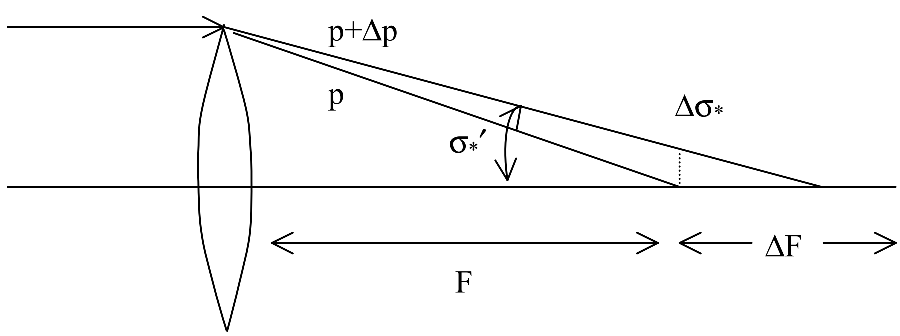


Figure 2.20

$$\Delta\sigma_* = (\Delta F)\sigma_*' = F\frac{\Delta F}{F}\sigma_*' = F\delta\sigma_*'$$

$$\therefore \frac{\Delta\sigma_*}{\sigma_*} = \frac{F}{\beta_*}\delta \tag{2.44}$$

For the NLC; F = 1 m, β^* = 0.2 mm, $\delta = 2 \times 10^{-3}$, we find $\Delta\sigma_*/\sigma_* \approx 10$, i.e., a tenfold increase in the IP spotsize! Thus the momentum dependence of the spot size, sometimes referred to as the *chromaticity*, is a serious effect.

The chromaticity can be corrected by the scheme schematically illustrated in the following figure:

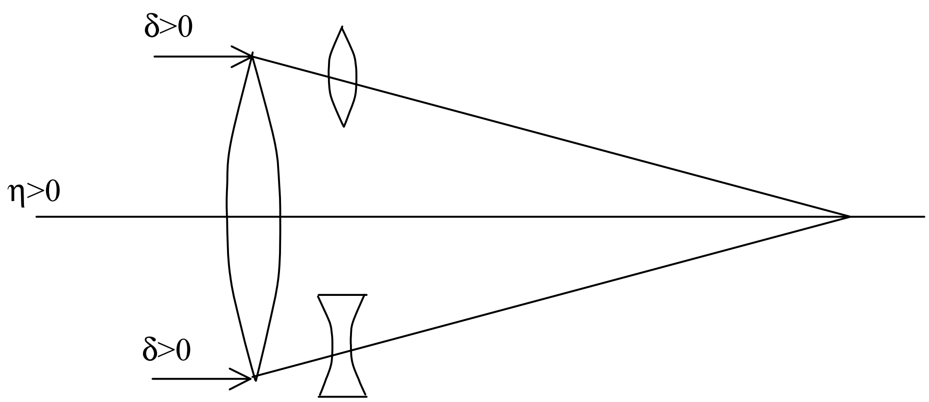


Figure 2.21

First a dispersion is created upstream of the lens so that the beam at the lens is spread in X according to the momentum δ. The δ-dependent focusing of the main lens can then be corrected by an x-dependent focusing indicated by a pair of positive and negative lens located at $x > 0$ and $x < 0$, respectively. In practice the x-dependent focusing is accomplished by a sextupole lens, in which the angular deflection is given by

$$\Delta x' = hx^2. \tag{2.45}$$

The sextupole is an example of non-linear elements which could cause resonance in the betatron motion.

2.2.9 Linear and non-linear resonances

The equation of the betatron motion including errors in linear and non-linear terms located at s=0 is

$$\frac{d^2x}{ds^2} + K(s)x = \delta(s)\sum_n a_n x^n. \tag{2.46}$$

By using the Floque variable, and by introducing a Fourier analysis of the s-dependence, the equation becomes

$$\frac{d^2u}{d\theta^2} + \nu^2 u = \sum_{m,n} b_{mn} e^{\pm im\theta} u^n$$

Here b_{mn}'s are constants depending on the envelope function and the strength of the errors.

The unperturbed solution neglecting the RHS is $u = e^{\pm i\nu\theta}$. The RHS then becomes a sum of terms $e^{\pm im\theta} e^{\pm in\nu\theta}$. If some of these terms become of the form $e^{\pm i\nu\theta}$ then the equation becomes a resonantly driven harmonic oscillator. The condition for resonance is

$$\pm M\nu \pm N = 0$$

where M, N are the integers.

For complete transverse the motion in both x- and y-directions, it can be shown that the resonance blow-up occurs if

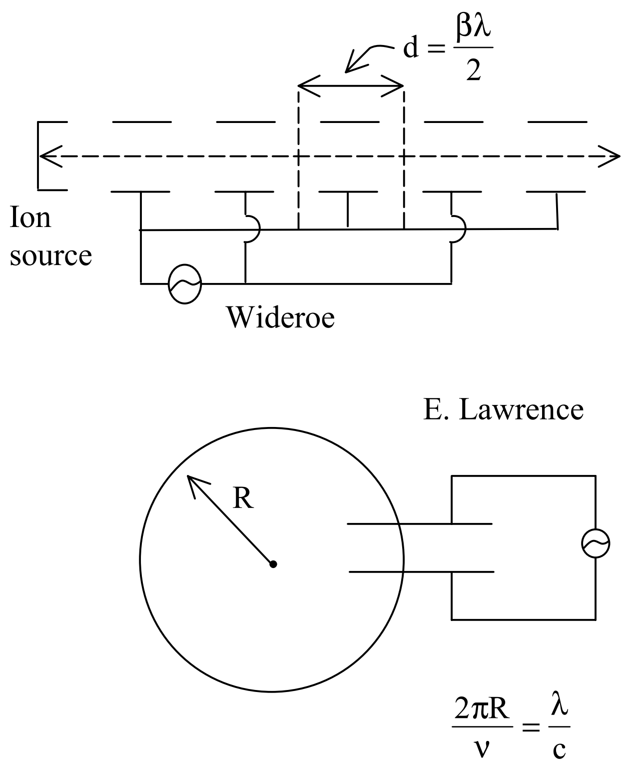
$$M\nu_x + N\nu_y = P, \tag{2.47}$$

where M, N, and P are all non-vanishing positive integers. (If M and N are of opposite sign, then the motions are coupled but stable.)

The linear and non-linear errors producing the resonances must be minimized and the tune must be chosen away from the resonance conditions as much as possible.

2.3 Acceleration and longitudinal motion

Electrostatic acceleration is limited due to HV breakdown. Oscillating voltage can be arranged so that a particle can be repeatedly accelerated (Ising 1924, Wideroe 1928, Lawrence 1930, Alvarez 1945, …)



Resonance acceleration → particle motion and the oscillating field are in resonance.

Figure 2.22

Modern accelerators utilize the rf structures and sources developed during WWII. The simplest of the rf accelerating structure is the cylindrical cavity.

2.3.1 Cylindrical cavity (pill box)

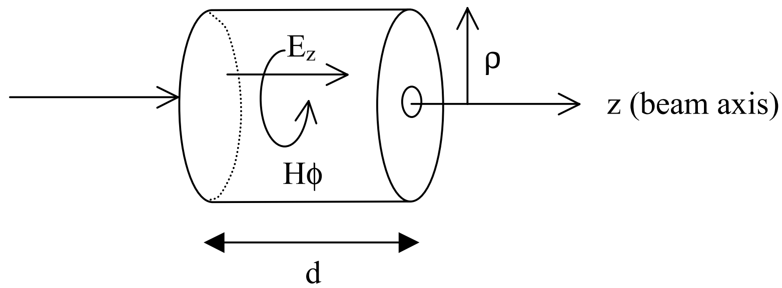


Figure 2.23

The cavity can support many (∞) modes, from which a desired mode can be chosen by exciting the cavity with a correct frequency. The simplest mode useful for acceleration is TM_{010} mode (TM: transverse magnetic, 0 → no ϕ-variation, 1 → first radial mode, 0 → no z-variation), with frequency $\omega = 2.405\ c/\rho$. The z-component electric field is (assume the perturbation due to the small beam hole is negligible)

$$\varepsilon_z = \mathbf{E}_o J_o\left(2.405\frac{r}{\rho}\right)\cos(\omega t + \varphi_o) \tag{2.48}$$

The energy gain of a particle passing the center of the cavity at t=0 is

$$\Delta E = e\int_{d/2}^{d/2} \varepsilon_z dz = e\mathbf{E}_o \int_{-d/2}^{d/2} \cos\left(\frac{\omega z k}{v} + \varphi_o\right) dz$$

$$\therefore \Delta E = eV\left(\frac{\sin\theta}{\theta}\right)\cos\varphi_o, \quad \theta = \frac{\omega}{2v} d\,. \tag{2.49}$$

Here $V = \mathbf{E}_o d$ is the maximum voltage for static case. Clearly, the phase φ_o should be 0 for maximum acceleration. The factor $(\sin\theta/\theta)$ is known as the transit time factor, representing the reduction in the effective acceleration voltage due to the changing rf field while the particle is being accelerated. A reasonable choice would be $\theta = \pi/2$ corresponding to $d = \lambda/2$, where λ is the rf wavelength, $\lambda = 2\pi c/\omega$. (We assume that the particles are relativistic $v \approx c$).

To maintain the accelerating field the cavity must be fed with rf power to balance the ohmic loss at the cavity surface from the oscillating current. The power P_{loss} is proportional to the voltage squared:

$$P_{loss} = \frac{V^2}{R_a}. \tag{2.50}$$

The quantity R_a is known as the *shunt impedance*. You want a large R_a so that the required power for a given acceleration voltage is small. Note that the shunt impedance is *inversely* proportional to the surface resistance.

In a long linac it is more useful to write Eq. (2.50) in terms of quantities per unit distance as follows:

$$P_{loss}/L = \frac{(V/L)^2}{(R_a/L)}$$

Here P_{loss}/L is the power loss per unit distance along the linac, and similarly for V/L and R_a/L. For room temperature structures (using copper as conducting material) R_a/L is proportional to $\sqrt{\omega}$. Thus it is advantageous to employ higher frequency rf such as the x-band ($\omega = 11.4$ GHz) for the NLC. The drawback is the fact that the structure becomes small and the wakefield effect becomes more severe.

Superconducting rf operating at 2K (the liquid He temperature) is attractive because the shunt impedance, being proportional to Q, is about 10^6 ($\approx 10^{10}/10^4$) times larger compared to the room temperature structures. The rf power dissipated as heat at 2K becomes negligible. However, the cryogenic system removing the heat at room temperature becomes a non-trivial addition to the complexity and cost.

- **Exercise:** Discuss the beam power and rf power for NLC and TESLA.

2.3.2 Multi-cell cavities

The total acceleration can be doubled by two weakly coupled $d = \lambda/2$ pill boxes:

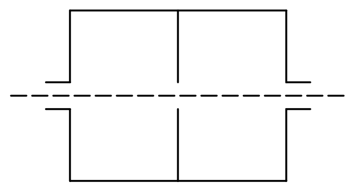


Figure 2.24

In general such a cavity supports two modes of oscillations, the 0-mode in which two cells are in phase and the π-mode in which the cells are out of phase. In the 0-th mode the structure is equivalent to a $d = \lambda$ cavity for which the transit time factor vanishes. In the π-mode the acceleration doubles since the phase changes to 180° while the particles pass through the first cavity.

For N>2 cells, there are N modes corresponding to N possible phase shifts per cavity, $2\pi m/N$, $m = 0,1,N-1$. A desired mode can be chosen by selecting the correct rf frequency.

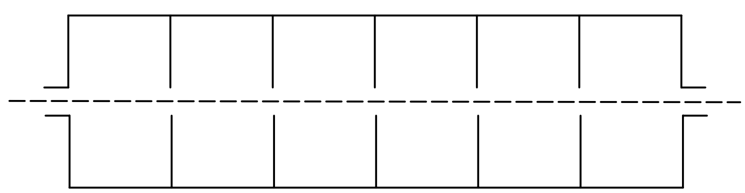


Figure 2.25

This structure can also be viewed as a smooth waveguide loaded by a set of diaphragms. The loading is necessary to reduce the phase velocity, which is faster than c in a smooth waveguide, to the particle velocity $v \approx c$ for resonant acceleration.

2.3.3 Acceleration in linear accelerators (linacs)

In a linac structure discussed above, the accelerating field can be represented by a traveling sinusoidal wave

$$\varepsilon_z = \mathbf{E}_o \cos(\omega t - kz) \tag{2.51}$$

where the phase velocity $\omega/k = c$. A particle entering $z = 0$ at $t = t_o$ sees a constant phase

$$\varphi = \omega\left(\frac{z}{c} + t_o\right) - kz = \omega t_o . \tag{2.52}$$

The energy gain in a length L is

$$\Delta E = e\int_0^L \varepsilon_z dz = e\mathbf{E}_o L\cos(\omega t_o)$$

$$\therefore \gamma(L) = \gamma(0) + \frac{e\mathbf{E}_o L}{mc^2}\cos(\omega t_o) \tag{2.53}$$

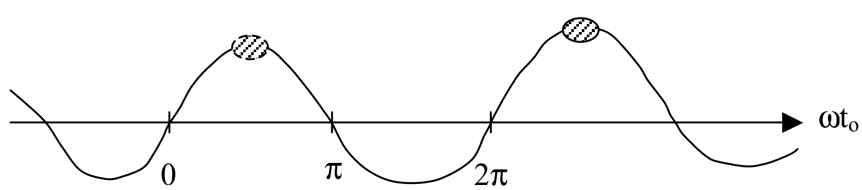


Figure 2.26

Particles are accelerated for $0 < \omega t_o < \pi$ and similar intervals separated by 2π. Thus electron beams must be in bunches of length $\Delta z < \lambda/2$ (λ = rf wavelength) separated by a multiple of rf wavelengths. For optimum acceleration ($\Delta z < \lambda/2$) bunches should be short and placed at the crest of the rf waveform. The energy spread in the beam is

$$\frac{\Delta\gamma}{\gamma} = 1 - \cos(2\pi\Delta z/2\lambda) \approx \frac{1}{2}(\pi\Delta z/\lambda)^2. \tag{2.54}$$

Exercise: Discuss bunch length and energy spread for the NLC X band.

It is sometimes desirable that there is a linear variation of energy along the length of the bunch. Thus, the BNS damping discussed in the next subsection requires that the energy of the electrons in the tail of the bunch is less than those in the head. This can be arranged by placing the bunches off the rf crests. However, it will be necessary to provide more rf power to achieve the same overall acceleration.

2.3.4 Adiabatic damping

Emittance is conserved for transverse motion when there is no acceleration. With acceleration the transverse angle becomes smaller:

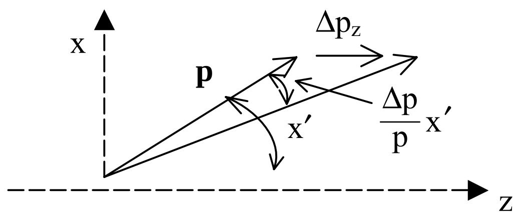


Figure 2.27

Thus, the transverse emittance will not be conserved. However, the canonical phase space area ($\Delta x \Delta p_x$) will be conserved. Since $\Delta p_x = m\gamma\beta\Delta x'$, the quantity referred to as the *normalized* emittance

$$\varepsilon_{nx} = \gamma\beta\varepsilon_x \approx \gamma\varepsilon_x \tag{2.55}$$

is conserved, where ε_x is the *unnormalized* emittance inroduced in previous section. As the energy increases due to acceleration the unnormalzied emittance decreases as

$$\varepsilon_x = \varepsilon_{nx}/\gamma. \tag{2.56}$$

This phenomenon is referred to as the adiabatic damping.

Electron beams for linear accelerators are produced by thermionic guns and then prepared for the appropriate bunch trains by going through a sequence of bunchers. The emittances of the bunched beams entering the linacs are too large for high-brightness applications such as free-electron lasers and linear colliders. Therefore *damping rings* are used to reduce the emittance to a desired level before entering the main linac.

Recently another type of gun, *rf photocathode gun* driven by high-power lasers, has been developed. In this gun, bunched beams are generated by laser pulses hitting the photocathode surface and are accelerated by the strong rf field. The normalized emittance from the rf photocathode gun is small, about a few times 10^{-6} m-rad, which is about the same as the emittance at damping rings. The rf photocathode gun therefore obviates the damping rings for the electron beams. However, damping rings are still necessary to achieve a small vertical emittance and also for positrons.

2.3.5 Wakefield and instabilities

Passage of charged particle beams induce electromagnetic field in rf cavities and other structures present in accelerators. The beam-induced fields, the *wakefield*, act back on the beams and may cause instabilities.

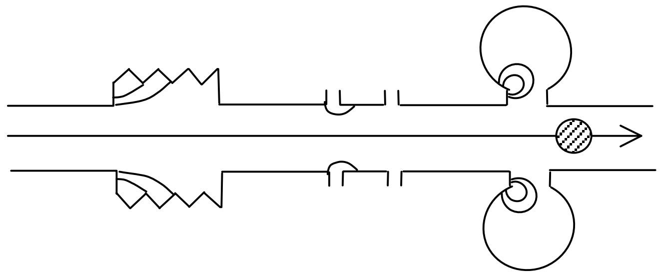


Figure 2.28

Longitudinal wakefields may lead to an energy spread in the beam. Transverse wakefield may cause a blow-up in the betatron motion leading beam breakup (BBU).

Wakefields are characterized by a wakefunctions which give the force on a test charge following an exciting charge at a distance z.

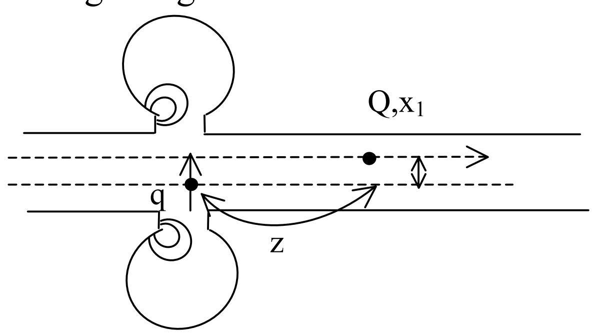


Figure 2.29

For example, the transverse wakefield function $W_1(z)$ is defined via the following relationship:

Force (energy per unit distance) on a test charge

$$= qQ\, W_1\,(z)\, x_1$$

The transverse beam breakup (BBU) instability can be understood by modeling the bunch as two equal short bunches separated longitudinally by a distance σ_z.

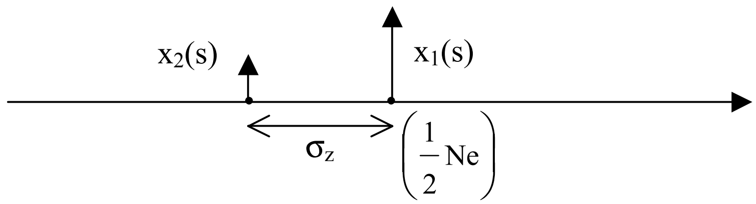


Figure 2.30

The head bunch undergoes a free betatron oscillation. Assuming a constant focusing, $K(s) = k_\beta^2$ in Eq. (2.20),

$$x_1(s) = \hat{x}_1 \cos(k_\beta s). \tag{2.57}$$

The equation for the displacement x_2 of an electron in the trailing part is

$$\frac{d^2 x_2}{ds^2} + k_\beta^2 x_2 = \frac{Ne^2 W_1(z)\hat{x}_1 \cos(k_\beta s)}{2E} \tag{2.58}$$

Here N = # of electrons, E = electron energy, e = electron charge. The RHS is the influence of the wakefield generated by the leading part. Eq. (2.58) has a solution

$$x_2(s) = \frac{Ne^2 W_1(z)\hat{x}_1}{4k_\beta E} s \sin(k_\beta s) \tag{2.59}$$

The betatron amplitude of the electrons in the trailing part grows linearly and will break out of the bunch. The amplification factor Υ over distance s = L may be defined as follows:

$$\Upsilon = \frac{Ne^2 W_1(z) L}{4k_\beta E} \tag{2.60}$$

For the case of the SLAC linac, taking z = rms bunchlength = 1 mm, $W_1(z) = 1.8$ V/(pC)(mm)(m), $k_\beta = 6 \times 10^{-5}$ (mm^{-1}). For a bunch charge Ne = 8 nC, the magnification factor of 1 GeV electron over the length s = 3 km is $\Upsilon = 180$!

The wakefield is a strong function of the iris radius a of the accelerating structure. For example the transverse wavefunction W_1 scales as a^{-3}. Thus the tolerance problem becomes much more challenging at high rf frequency such as the X-band.

The transverse BBU can be suppressed by arranging the focusing of the trailing part to be slightly stronger, i.e., by replacing $k_\beta^{\,2}$ in Eq. (2.58) by $(k_\beta + \Delta k_\beta)^2$ with

$$\Delta k_\beta = \frac{Ne^2 W_1(z)}{4Ek_\beta}. \qquad (2.61)$$

Under this condition both parts of the bunch move together and the BBU is suppressed. The focusing in the trailing part can in turn be made stronger by having it accelerated a little less than the head part. This method of curing the BBU instability is known as the BNS damping according to the inventors (V. Balakin, A. Novokhatsky, and V. Smirnov, 1983).

Wakefields can persist in the cavities for a long time to influence the motion of the bunches following the driving bunch, leading to multi-bunch BBU effects. The multibunch BBU can be partially cured by designing the cavities to have a small spread in frequencies.

2.3.6 Longitudinal motion in electron storage rings

We discuss the electron storage rings as a specific example of circular accelerators. In an electron storage ring, electron orbits are bent by dipoles to form closed loops. Electrons lose energy to the synchrotron radiation emitted during the bending motion, and regain energy in the rf cavity.

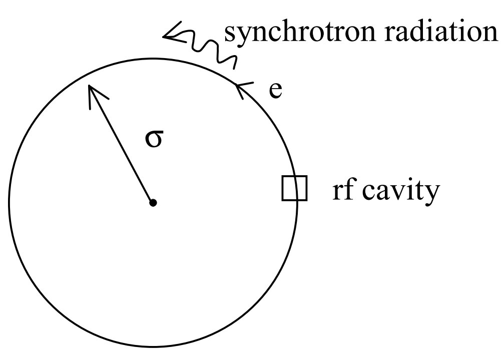


Figure 2.31

The circle in the above figure is meant to be not a real circle but a curve consisting of short arcs connected by straight lines. The straight lines represent *straight sections*, which are useful for many purposes: beam injection, rf station, place for collision point for high energy physics machines, or place for insertion devices for synchrotron radiation facilities, etc.

A *synchronous* electron with energy E_s enters the rf cavity at the phase ϕ_S to exactly balance the synchrotron radiation loss. The rf frequency ω_{rf} is an integer multiple of the orbit frequency ω of the synchronous electron so that the phase ϕ_S is the same turn by turn:

$$\omega_{RF} = h\omega_S, \text{ h = harmonic number.} \tag{2.62}$$

The orbit frequency varies in general with the energy of the electron due to the dispersion effect. Thus an off-energy electron with $E = E_S + \Delta E$ will see an rf phase ϕ_n at nth turn which is different from ϕ_s. The electron energy will then change slowly since the energy is not balanced.

The equations describing the turn by turn evolution are

$$\begin{aligned} E_{n+1} &= E_n + eV(\cos\phi_n - \cos\phi_s) \\ \phi_{n+1} &= \phi_n + 2\pi h\,\alpha_p \frac{E_n - E_s}{E_n} \end{aligned} \tag{2.63}$$

The first equation is easy to understand, with eV the peak acceleration voltage. Note that $eV\cos\phi_s$ is the synchrotron radiation loss. The second equation comes about because the variation in the orbit period is

$$\frac{\Delta T}{T} = \alpha_p \frac{\Delta E}{E} \tag{2.64}$$

Here α_p is known as the *momentum compaction factor*.

For a slow evolution, the difference equation can be written in the differential form:

$$\frac{d}{dt}(\Delta E) = \frac{eV}{T}(\cos\phi - \cos\phi_s)$$

$$\frac{d}{dt}\phi = \alpha_p \omega_{rf} \frac{\Delta E}{E}. \tag{2.65}$$

(Note: $2\pi h/T = \omega_{rf}$)

The equation for the phase excursion $\Delta\phi = \phi - \phi_s$ is

$$\frac{d^2}{dt^2}\Delta\phi = \frac{\alpha_p \omega_{rf} eV}{E_s T}\left(\cos(\phi_s + \Delta\phi) - \cos\phi_s\right). \tag{2.66}$$

This equation is in the same form as the equation of a pendulum in gravitational field, and is therefore referred to as the *pendulum equation*. If $\sin\phi_s > 0$ the motion for the small $\Delta\phi$ becomes a stable harmonic motion:

$$\frac{d^2}{dt^2}\Delta\phi = -\Omega_s{}^2\Delta\phi, \tag{2.67}$$

where

$$\Omega_s{}^2 = \frac{\alpha_p \omega_{rf} eV \sin\phi_s}{E_s T}. \tag{2.68}$$

The quantity Ω_s is known as the *synchrotron* frequency. The motion is unstable if $\sin\phi_s < 0$. The betatron tune ν_x and ν_y are usually of the order 10, while the synchrotron tune $\nu_s = \Omega_s/\omega_{rev} = \Omega_s T/2\pi$, is usually about 0.01-0.001.

The qualitative feature of the motion can be understood from

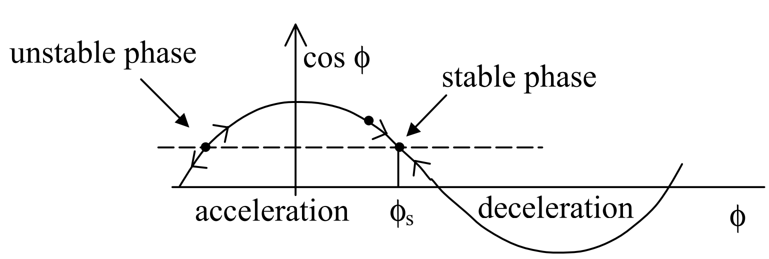


Figure 2.32

The principle of the *phase stability* was discovered by V.I. Veksler (1944) and E.M. McMillan (1945), and established that particles other than the exact synchronous one can be accelerated.

It is useful to introduce the Hamiltonian function corresponding to the longitudinal motion as follows:

$$H = \frac{\alpha_p \omega}{2} \frac{(\Delta E)^2}{E} - \frac{eV}{T}\left(\sin\phi - \phi\cos\phi_s\right) \tag{2.69}$$

The equation of motion is then

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial \Delta E}, \quad \frac{d\Delta E}{dt} = -\frac{\partial H}{\partial \phi}. \tag{2.70}$$

The phase space trajectories of the motion are obtained easily by finding curves that satisfy H = const. The trajectories are illustrated in Fig. 2.33. Both the small oscillations near the stable fixed point $\phi=\phi_s$ and the large amplitude oscillations are shown. For a large excursion, the stable motion is bounded by a *separatrix* in $(\Delta\phi,\Delta E)$ phase space. The stable area inside the separatrix is called the *bucket*. The phase space curve is periodic in ϕ with a period 2π. There are h buckets in the ring.

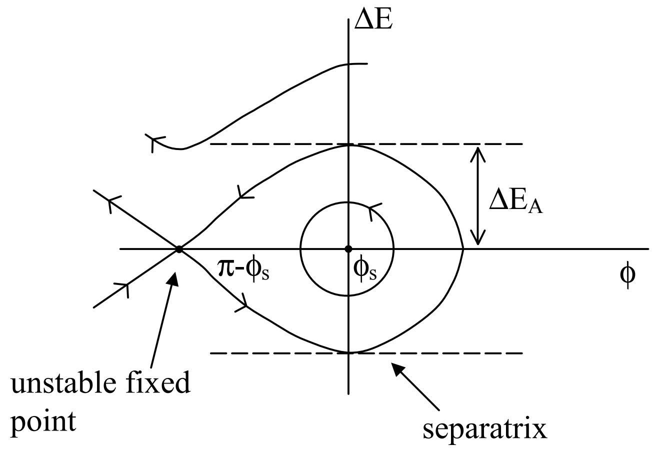


Figure 2.33

In electron storage ring the beam bunches are damped down due to the synchrotron radiation to a small region around a stable phase ϕ_s. The height $(\Delta E)_A$ of the separatrix is referred to as *the energy aperture* of the ring for obvious reason. The relative energy aperture $(\Delta E_A)/E$ is typically about a few percent. Electrons scattered into an energy beyond the energy aperture, for example via the *intra-beam scattering*, will be lost from the machine.

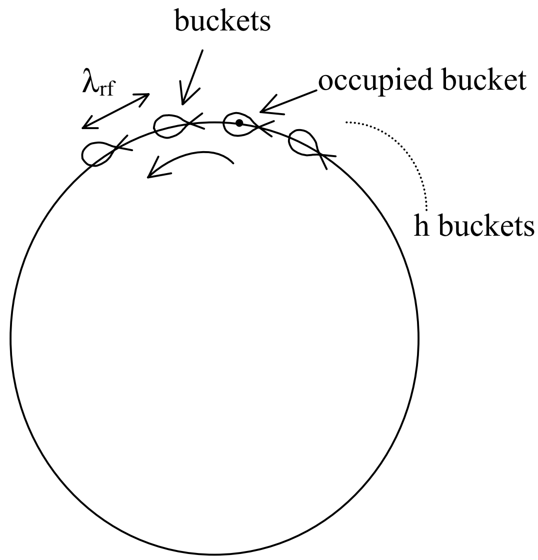


Figure 2.34

2.3.7 Emittance changing process: radiation damping and excitation

Unlike the case of motion under external forces, processes that distinguish individual particles do not conserve the emittance. An example of such process is the emission *synchrotron radiation* by relativistic electrons on curved trajectories.

Let us review some basic facts of the synchrotron radiation. The radiation is emitted in a cone of angular width $\Delta\theta \sim \gamma^{-1}$, with a typical frequency

$$\omega_c \approx \gamma^3 c/\rho, \tag{2.71}$$

where ρ is the radius of the curvature. There are about $\alpha = 1/137$ photons generated while the trajectory angle changes by γ^{-1}. Thus the number of photons per unit deflection angle is

$$\frac{dn_\gamma}{d\theta} \approx \alpha\gamma \tag{2.72}$$

The synchrotron radiation power is therefore

$$P_\gamma \approx \hbar\omega_c \frac{dn_\gamma}{d\theta}\frac{d\theta}{dt} \approx (\hbar\omega_c)\left(\frac{dn_\gamma}{d\theta}\right)\left(\frac{c}{\rho}\right) = \frac{e^2 c\gamma^4}{\rho}. \tag{2.73}$$

For a simple discussion of emittance changing processes, let us assume that the electron focusing is constant (Section 2.2.3). The emittance change $\Delta\varepsilon_x$ due to a change in x and x′ can be evaluated as follows:

$$\begin{aligned} \Delta{\varepsilon_x}^2 &= 2\varepsilon_x\Delta\varepsilon_x = \Delta\left(\left\langle x^2\right\rangle\left\langle x'^2\right\rangle\right) \\ &= \frac{\varepsilon_x}{\beta_x}\Delta\left\langle x^2\right\rangle + \beta_x\varepsilon_x\Delta\left\langle x'^2\right\rangle \end{aligned} \tag{2.74}$$

Here we have assumed $\langle xx'\rangle = 0$, and used $\langle x^2\rangle = \beta_x\varepsilon_x$, $\langle x'^2\rangle = \varepsilon_x/\beta_x$. It then follows:

$$\Delta\varepsilon_x = \frac{1}{2}\Delta\left(\frac{\left\langle x^2\right\rangle}{\beta_x} + \beta_x\left\langle x'^2\right\rangle\right) \tag{2.75}$$

An electron moving on a circular trajectory emits synchrotron radiation in the direction parallel to the instantaneous moment *p*. The electron momentum is thereby reduced by Δp. The electron is then accelerated in the rf cavities to recover the energy loss. However the acceleration is in the z-direction, thus the electron angle becomes reduced as is clear from the following diagram.

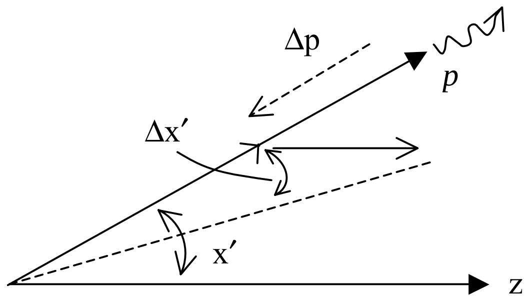


Figure 2.35

We have

$$\Delta x' = -\frac{\Delta p x'}{p} = -x'\frac{\Delta E}{E}. \tag{2.76}$$

The change in the emittance is (x does not change)

$$\Delta\varepsilon_x = \beta_x \langle x'\Delta x'\rangle = -\beta_x \langle x'\rangle^2 \frac{\Delta E}{E},$$
$$\therefore \left.\frac{d\varepsilon_x}{dt}\right|_D = -\varepsilon_x \frac{P_\gamma}{E} \tag{2.77}$$

Here $P_\gamma = \Delta E/\Delta t$ is the synchrotron radiation power. Let's introduce the time

$$\tau_D = E/P_\gamma, \tag{2.78}$$

which may be interpreted as the time to radiate away electron's kinetic energy to synchrotron radiation. The emittance evolution is

$$\varepsilon_x = \varepsilon_x(0)e^{-t/\tau} \tag{2.79}$$

The time τ is known as the radiation damping time.

Synchrotron radiation is emitted in discrete quanta of photons, leading to the heating of the beam. To estimate the heating effect, we note that since the electron energy changes during the emission but not the transverse coordinate x, the betatron oscillation coordinate x_β changes according to Eq. (2.39)

$$\Delta x_\beta = -\eta_x \frac{\Delta E}{E} \tag{2.80}$$

The change in the emittance becomes

$$\begin{aligned} \Delta\varepsilon_x &= \frac{1}{\beta_x}\left(\left\langle (x_\beta + \Delta x_\beta)\right\rangle^2 - \left\langle x_\beta{}^2 \right\rangle\right) \\ &= \frac{1}{\beta_x}\left(\left\langle 2x_\beta \Delta x_\beta \right\rangle + \left\langle (\Delta x_\beta)^2 \right\rangle\right) \end{aligned} \tag{2.81}$$

The first term in the above vanishes after averaging. The next term, which is of the second order and positive.

$$\Delta\varepsilon_x = \frac{\eta_x{}^2}{\beta_x}\frac{\left\langle (\Delta E)^2 \right\rangle}{(E)^2} \approx \frac{\eta_x{}^2}{\beta_x}\frac{(\hbar\omega_c)^2}{E^2} \tag{2.82}$$

Here $\hbar\omega_c$ is the average energy of a synchrotron radiation photon. The heating rate can be obtained by multiplying the above equation by $dn_\gamma/dt = P_\gamma/\hbar\omega_c$ = # of photons emitted per unit time:

$$\left.\frac{d\varepsilon_x}{dt}\right|_H = \frac{\eta_x{}^2}{\beta_x}\frac{(\hbar\omega_c)^2}{E^2}\frac{dn_\gamma}{dt} \approx \frac{\eta_x{}^2}{\beta_x}\frac{\hbar\omega_c}{E^2}P_\gamma > 0 \tag{2.83}$$

The emittance will approach equilibrium value ε_{xo} determined by the condition that the total rate vanishes:

$$\frac{d}{dt}\varepsilon_x = \left(\frac{d}{dt}\varepsilon_x\right)_D + \left(\frac{d}{dt}\varepsilon_x\right)_H = -\varepsilon_x \frac{P_\gamma}{E} + \frac{\eta_x{}^2}{\beta_x}\frac{\hbar\omega_c}{E^2}P_\gamma \tag{2.84}$$

$$\therefore \varepsilon_{xo} = \frac{\eta_x{}^2}{\beta_x} \frac{\hbar\omega_c}{E}.$$

Using the constant focusing case, $\eta_x \approx \beta_x{}^2/\rho$, and $\omega_c \approx \hbar\gamma^3 c/\rho$,

$$\varepsilon_{xo} \approx \left(\frac{\beta}{\rho}\right)^3 \gamma^2 \lambda\!\!\!^{-}_c \tag{2.85}$$

where $\lambda\!\!\!^{-}_c = \hbar/mc = 3.8 \times 10^{-13}$ m is the Compton wavelength.

In the vertical direction the emittance can in principle damp down to zero since there are no excitation mechanisms. The vertical emittances in practical machines do not vanish due to coupling of the horizontal motion via *skew quadrupoles*.

Modern synchrotron radiation facilities are based on high-brightness electron storage rings where the damping effects are optimized for small emittances. *Damping rings* are similar to the synchrotron radiation storage rings. They are a crucial component of the accelerator complex in linear colliders.

2.3.8 Oide effect

The achievable small beam size in linear colliders could be limited due to synchrotron radiation in the FF (final focus) lens as was pointed out by Oide. In Figure 2.36, the trajectory of an electron entering the FF lens with a typical transverse displacement σ is shown to be deflected by an angle σ_*' to be focused at the IR point. For a given emittance, the spotsize σ_* at IR is minimized by minimizing β^*, thus maximizing the angle σ_*'.

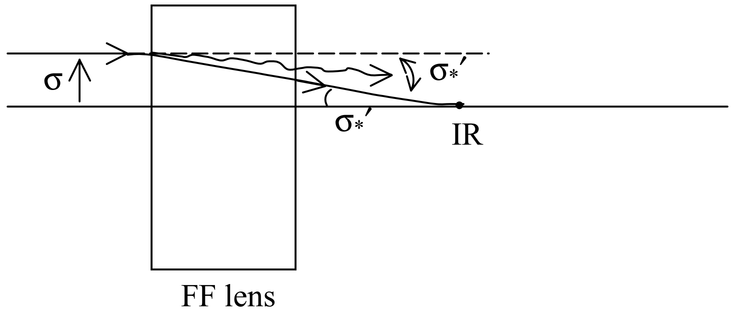


Figure 2.36

However, more synchrotron radiation photons are produced as the deflection angle is increased in the lens. The beam size increase due to the photon emission in the FF lens can become significant and offset the beam size decrease for a sufficiently small β^*.