(ANL & UofC) Effort on Ionization Cooling Theory

Kwang-Je Kim
University of Chicago and Argonne National Laboratory

MUTAC Review
Lawrence Berkeley National Laboratory
October 18-19, 2001

ANL & UofC Effort on Ionization Cooling Theory

- **KJK & CXW**
- **Papers:**
  - **Formulas for transverse ionization cooling in SFC**

    PRL 85(4) 700, 2000 (KJK & CXW)
  - **Recursive solution for beam dynamics study of ICC**

    PRE 64(5), 2001 (CXW & KJK)
  - **Beam parameterization and invariants in a period SC (CXW & KJK)**
  - **Magnetic field expansion for particle tracking in a bent-solenoidel channel**

    (CXW & Lee C. Teng)
  - **Linear theory of transverse & longitudinal IC in quadrupole channel**

    (CXW & KJK)
  - ***Linear theory of 6-D ionization cooling***

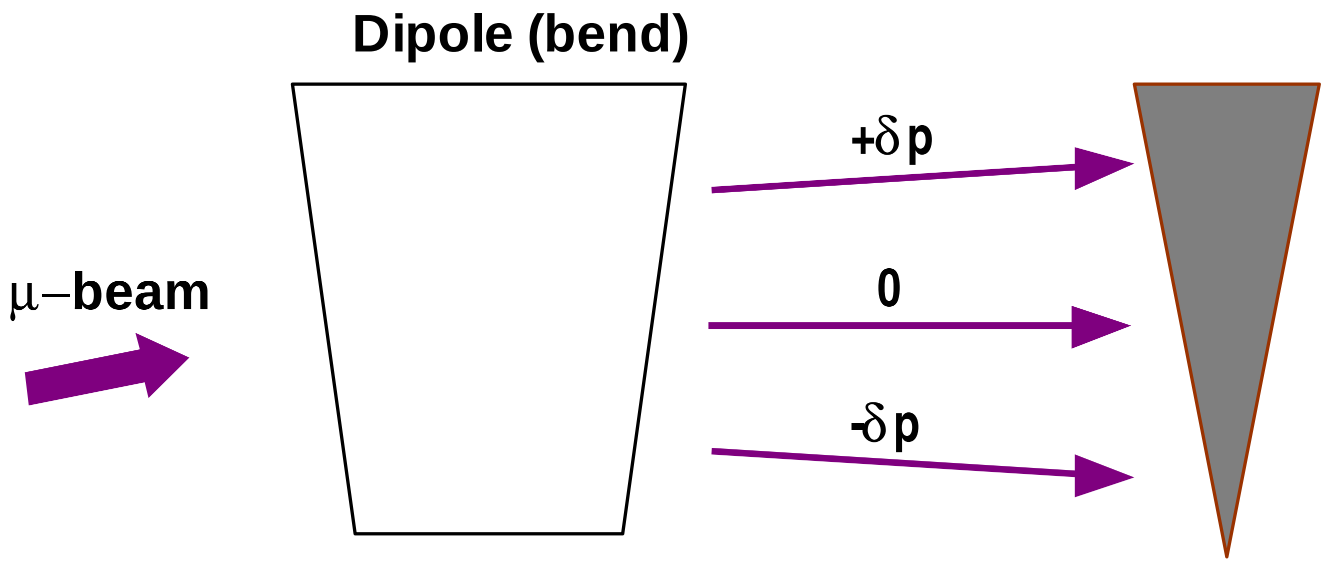
    Snowmass Proceedings – to be published

Ionization Cooling Theory in Linear Approximation

- **Similar in principle to radiation damping in electron storage rings, but needs to take into account:**
  - **Solenoidal focusing and angular momentum**
  - **Emittance exchange**

- ***Messy unless guided by fundamental principles!***
  - **Orthonormal basis of moments for Hamiltonian system,**
  - **Dissipation and fluctuation leading to an equilibrium**
  - **Slow evolution near equilibrium can be described by five Hamiltonian invariants**

Emittance Exchange



Dipole introduces dispersion (η)

$x \rightarrow x_0 + \eta\, \delta p/p$

Wedge Absorber reduces energy spread

Hamiltonian Under Consideration

Solenoid + dipole + quadrupole + RF + absorber

Goal: theoretical framework and possible solution

Lab frame

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \underbrace{\frac{1}{2}\kappa(s)^2\left(x^2+y^2\right) + \kappa(s)\left(xp_y - yp_x\right)}_{\text{solenoid}}$$

$$\underbrace{-\frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2}}_{\text{dipole}} \underbrace{- \frac{1}{2}k(s)\left(x^2-y^2\right)}_{\text{quadrupole}} \underbrace{+ \frac{1}{2}\left[\delta^2 + v(s)z^2\right]}_{\text{r.f.}}$$

rotating frame with **symmetric focusing**

$$\tilde{H} = \frac{1}{2}\left(\tilde{p}_x^2 + \tilde{p}_y^2\right) + \frac{1}{2}K(s)\left(\tilde{x}^2+\tilde{y}^2\right) - \frac{\tilde{x}\delta\cos[\theta(s)]}{\rho(s)} - \frac{\tilde{y}\delta\sin[\theta(s)]}{\rho(s)}$$

$$K(s) = \kappa(s)^2 + \frac{1}{2\rho(s)^2}, \quad \theta(s) = -\int_0^s \kappa(\bar{s})\,d\bar{s}$$

Model for Ionization Process in Larmor Frame

Transverse:

$$\frac{d}{ds}\mathbf{P} = -\eta\left(\mathbf{P} - \kappa\mathbf{e}_z \times \mathbf{x}\right) + \xi$$

ξ → M.S.

Longitudinal:

$$\frac{d\delta}{ds} = -\bar{\eta} + \frac{\partial\eta}{\partial x}x + \frac{\partial\eta}{\partial y}y + \xi_\delta$$

$\frac{\partial\eta}{\partial x}$, $\frac{\partial\eta}{\partial y}$ → wedge

ξ_δ → straggling

$\bar{\eta}$: Average loss replenished by RF

$$(u, v) \equiv \left(\frac{\partial\eta}{\partial x}, \frac{\partial\eta}{\partial y}\right)$$

Equations for Dispersion Functions

In Larmor frame

Dispersion function decouples the betatron motion and dispersive effect

$$x = x_\beta + D_x(s)\delta \quad y = y_\beta + D_y(s)\delta$$

$$\tilde{D}_x'' + K(s)\tilde{D}_x = \frac{\cos[\theta(s)]}{\rho(s)}$$

$$\tilde{D}_y'' + K(s)\tilde{D}_y = \frac{\sin[\theta(s)]}{\rho(s)}$$

Equation for 6-D Phase Space Variables

- $x = x_\beta + \delta D_x, \quad P_x = P_{x\beta} + \delta \; D'_{(x\leftrightarrow y)}$
- $z = z_\beta - D'_x x + D_x P_x - D'_y y + D_y P_y$
- Dispersion vanishes at cavities $D_x \cdot V = D'_x \cdot V = 0$
- Drop suffix β

$$x' = P_x - D_x \delta'$$
$$P'_x = -\kappa^2 x - D'_x \delta' - \eta \overline{P_x} + \xi_x$$
$$y' = P_y - D_y \delta'$$
$$P'_y = -\kappa^2 P_y - D'_y \delta' - \eta \overline{P_y} + \xi_y$$
$$z' = I(s)\delta + \eta\left(D_x \overline{P}_x + D_y \overline{P}_y\right)$$
$$\delta' = -V(s)z - (ux + vy) - (u D_x + v D_y)\delta + \xi_\delta$$

$$\overline{P_x} = P_x + \kappa y + \delta\left(D'_x + \kappa D_y\right) - \xi_x$$
$$\overline{P_y} = P_y - \kappa x + \delta\left(D'_x - \chi P_x\right) - \xi_y$$

Equation for Moments

- **Phase space vector:**

$$\frac{d}{ds}X = (K - A)X + \Xi, \quad X = \begin{pmatrix} x \\ P_x \\ y \\ P_y \\ z \\ \delta \end{pmatrix}$$

K = JH (Hamiltonian motion)

A = Dissipation

Ξ = Fluctuation

- **Moment matrix**

$$R_{ij} = \langle x_i x_j \rangle \quad (6\times 6)$$

$$\frac{dR}{ds} = (K - A)\,R + R\left(K^T - A^T\right) + \chi$$

χ → excitation

Assume the non-Hamiltonian effects are small.

Moment Evolution Near Equilibrium

(F. Ruggiero, E. Picasso, & L.A. Radicati, Ann. of Physics 197, 396 (1990))

- **System approaches an equilibrium under damping & excitation**
- **Near the equilibrium, the moment matrix R changes slowly (weak dissipation)**
- **R can be expanded by a complete set of orthonormal basis matrices:**

$$\mathbf{R} = \sum_{\alpha=1}^{21} \varepsilon_\alpha(s)\, \sigma_\alpha(s)$$

- **σ_α are constructed from tensor products of the eigenvectors of the Hamiltonian motion. Most of σ_α vary rapidly**

$$\boldsymbol{\sigma}_{\boldsymbol{\alpha}}(s+\ell) = e^{i(\pm\mu_p \pm\mu_q)}\, \boldsymbol{\sigma}_{\boldsymbol{\alpha}}(s)$$

(ℓ = lattice period, μ_p = phrase advance per period (p = x, y, z)

- ***Coefficients ε_α of the rapidly varying σ_α are small and can be neglected***
- ***The system near equilibrium can be represented by a special set of matrices, with coefficients which are invariants of the Hamiltonian motion***

Hamiltonian Invariants

- **Three Courant-Snyder invariants:**

$$\varepsilon_x = \frac{1}{2}\left(\gamma\left\langle x^2\right\rangle + 2\alpha\left\langle xP_x\right\rangle + \beta\left\langle P_x{}^2\right\rangle\right), \ (x \leftrightarrow y)$$

$$\varepsilon_z = \frac{1}{2}\left(\gamma_z\left\langle z^2\right\rangle + 2\alpha_z\left\langle z\delta\right\rangle + \beta_z\left\langle \delta^2\right\rangle\right)$$

(γ, α, β), (γ_z, α_z , β_z); Twist parameters for $\perp$ and $\|$

- ***Two more invariants when $\mu_x = \mu_y$:***

$$\varepsilon_L = \frac{1}{\sqrt{2}}\left\langle x P_y - y P_x\right\rangle$$

$$\varepsilon_{xy} = \frac{1}{\sqrt{2}}\left(\gamma\left\langle x y\right\rangle + \alpha\left\langle x P_y + y P_x\right\rangle + \beta P_x P_y\right)$$

Hamiltonian Invariants (Contd.)

Inverse relationship leads to moment parametrization:

$$\left(\langle x\rangle^2, \langle x P_x\rangle, \langle {P_x}^2\rangle\right) = \varepsilon_x(\beta, -\alpha, \gamma), \ (x \leftrightarrow y \leftrightarrow z)$$

$$\left(\langle xy\rangle, \frac{\langle x P_y + y P_x\rangle}{2}, \langle P_x P_y\rangle\right) = \frac{\varepsilon_{xy}}{\sqrt{2}}(\beta, -\alpha, \gamma)$$

$$\langle x P_y - y P_x\rangle = \sqrt{2}\,\varepsilon_L$$

Evolution of Invariants Under Non-Hamiltonian Force

- ε_i, i = 1, …5, vary slowly due to dissipation and excitation
- Equation for ε_i is obtained by inserting the C-S parameterization into the moment equation
- The calculation is manageable since ε_α, $\alpha > 5$, are neglected in view of the RPR analysis!

IC and Emittance Exchange Equation in Linear Approximation

$$\frac{d\varepsilon_i}{ds} = \sum_{j=1}^{5} G_{ij}\,\varepsilon_j + \chi_i$$

G_{ij} :

	x	y	xy	L	z
x	$-\eta_x$	0	A	B	0
y	0	$-\eta_y$	A	B	0
xy	A	A	$-\eta_{xy}$	0	0
L	B	B	0	$-\eta_L$	0
z	0	0	0	0	$-\eta_z$

$$\eta_x = \eta \text{ - } u\,D_x,\ \eta_y = \eta \text{ - } u\,D_y,\ \eta_z = u\,D_x + v\,D_y,\ \ \eta_{xy} = \eta_L = \eta - \frac{1}{2}\left(u\,D_x + v\,D_y\right)$$

$$A = \frac{1}{2\sqrt{2}}\left(u\,D_y + v\,D_x\right),\ \ B = \frac{1}{\sqrt{2}}\,\kappa\beta\eta + \frac{1}{2\sqrt{2}}\left[\left(\alpha\,D_y + \beta\,D'_y\right)u - \left(\alpha\,D_x + \beta\,D'_y\right)v\right]$$

The Excitations

$$\chi_{\mathbf{x}} = \frac{1}{2} \mathcal{H}_{\mathbf{x}} \chi_\delta + \frac{\beta}{2} \chi$$

$$\chi_{\mathbf{y}} = \frac{1}{2} \mathcal{H}_{\mathbf{y}} \chi_\delta + \frac{\beta}{2} \chi$$

$$\chi_{\mathbf{z}} = \frac{1}{2} \left[\gamma_{\mathbf{z}} \left(\mathbf{D_x}^2 + \mathbf{D_y}^2 \right) \chi + \beta_{\mathbf{z}} \chi_\delta \right]$$

$$\chi_{\mathbf{xy}} = \frac{1}{\sqrt{2}} \mathcal{H}_{\mathbf{xy}} \chi_\delta$$

$$\chi_{\mathbf{L}} = \frac{1}{\sqrt{2}} \left(\mathbf{D_x D'_y} - \mathbf{D_y D'_x} \right) \chi_\delta$$

$$\mathcal{H}_{\mathbf{x}} = \gamma \mathbf{D_x}^2 + 2\alpha\, \mathbf{D_x D'_x} + \beta\, \mathbf{D'_x}^2, \; (\mathbf{x} \leftrightarrow \mathbf{y})$$

$$\mathcal{H}_{\mathbf{xy}} = \gamma \mathbf{D_x D_y} + \alpha \left(\mathbf{D_x D_{y'}} + \mathbf{D_y D_{x'}} \right) + \beta\, \mathbf{D'_x D'_y}$$

Remarks

- **The result generalizes the previous D = 0 analysis**

- **6-D phase spare area**

$$\varepsilon_{6D} = \varepsilon_z\left(\varepsilon_x\varepsilon_y - \frac{1}{2}\varepsilon_{xy}{}^2 - \frac{1}{2}\varepsilon_L{}^2\right)$$

$$\frac{d}{ds}\varepsilon_{6D} = -\left(\eta_x + \eta_y + \eta_z\right)\varepsilon_{6D} = -2\eta\,\varepsilon_{6D}$$

- **Choose** $u\,D_x = v\,D_y$

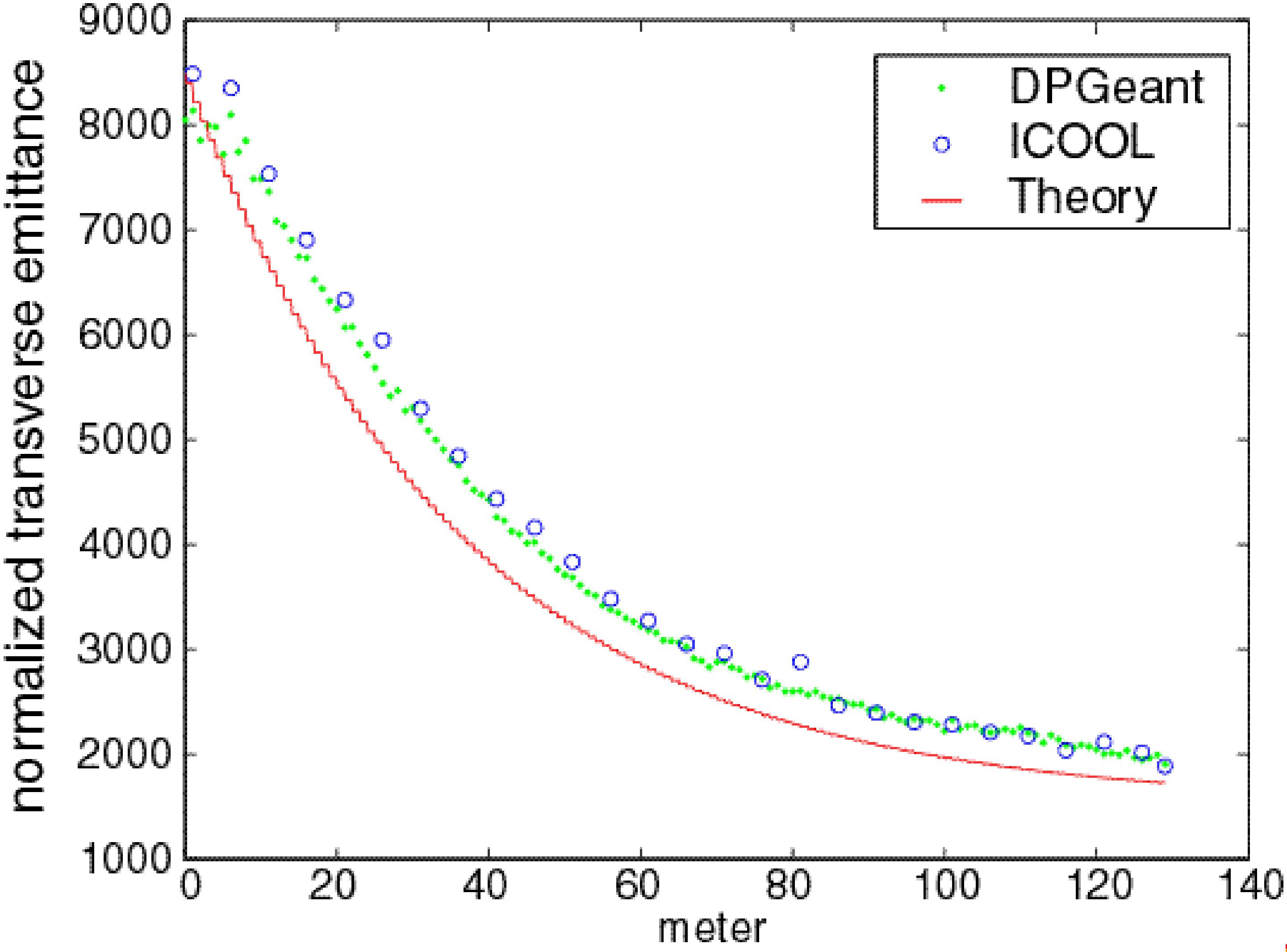
$$\Rightarrow \eta_x = \eta_y = \eta_{xy} = \eta_L = \eta_\perp$$

- **Averaged over one period,** $\chi_L = 0$

- **Design examples are being worked out**

Transverse Cooling Theory

D = 0 case



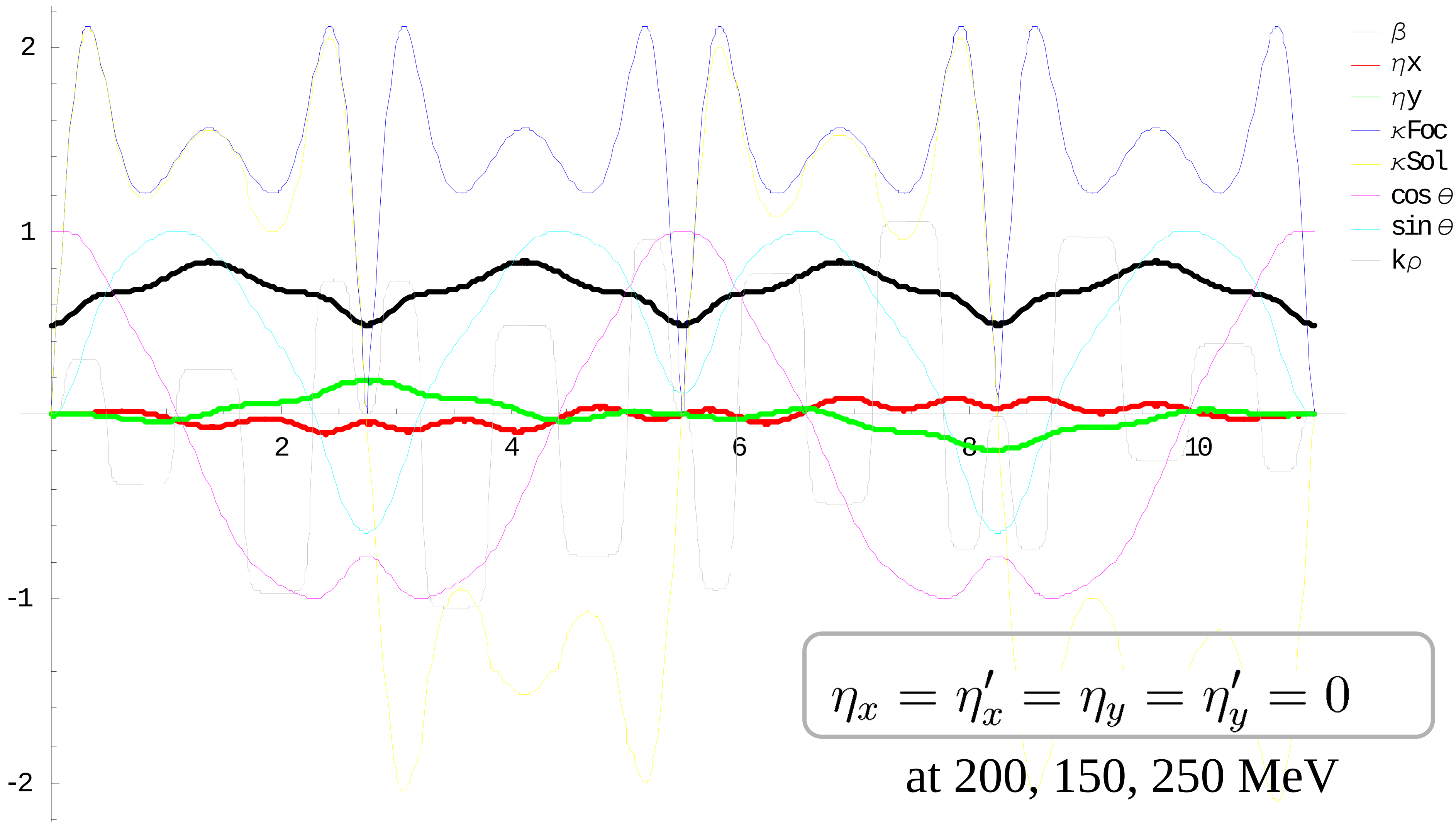
Design Principles

- Update good transverse cooling channels
- Maintain periodic structure, esp. beta function
- Create localized dispersion in desired periods

  — closed dispersion bump
- Maximum dispersion at minimum beta
- Keep symmetric focusing
- No dispersion in RF
- Dispersion section to be 1st order achromat

A Maxwell-Happy Solution

Max. dipole field ~ 0.7 T, dispersion ~ 20 cm



$$\eta_x = \eta'_x = \eta_y = \eta'_y = 0$$

at 200, 150, 250 MeV

A Maxwell-Happy Solution